REP NOTES FOR JACOB AND MATHILDE

Let k be an algebraically closed field of characteristic zero, and G a finite group. Last time we showed that $k[G]$ is a product of matrix algebras, and Daniel tried to give a "pure thought" proof that

$$k[G] \simeq \bigoplus_i \rho_i \boxtimes \rho_i^\vee$$

as a ring with $G \times G$-action, which ended up being a bit confusing; here the sum is over the irreps of G. (From the proof of the classification of semisimple algebras, we know this as a ring with left G-action, but we haven't pinned down the right G-action.) We'll start by proving this directly.

Goal. Write $k[G] = \oplus \rho_i \otimes V_i$, where $V_i = \mathrm{Hom}_G(\rho_i, k[G])$. We show that, on the isotypic block V_i corresponding to an irreducible ρ_i, the *right* G-action induces a representation isomorphic to V^*. Recall:

- The left G-action on V_i is trivial.
- The right G-action on $\rho_i \otimes V_i$ commutes with the left G-action, so it factors through
$$\rho \colon G \longrightarrow \mathrm{Aut}_G\big(V_i\big)$$
- We wish to identify this $\dim V$-dimensional representation of G with the dual representation $\rho^\vee$.

Matrix coefficients of V. Let V be an irreducible $k[G]$-module of dimension $n = \dim V$. Choose a basis $\{v_1, \ldots, v_n\}$ of V and the corresponding *dual basis* $\{v_1^*, \ldots, v_n^*\} \subset V^*$. For each pair (i,j) with $1 \le i, j \le n$, define the *matrix-coefficient function*

$$\phi_{i,j} \colon G \longrightarrow k, \quad \text{by} \quad \phi_{i,j}(g) \;=\; v_i^*\big(g \cdot v_j\big).$$

These functions belong to the group algebra $k[G]$, and we claim the subspace spanned by all $\{\phi_{i,j}\}$ for fixed V is precisely the isotypic piece of $k[G]$ corresponding to V. Equivalently, this is the sum of all copies of V in the left-regular representation.

Behavior under left and right actions.

- **Left action.** For $g' \in G$,
$$(g' \cdot \phi_{i,j})(g) \;=\; \phi_{i,j}\big(g'^{-1} g\big) \;=\; v_i^*\big(g'^{-1} g \cdot v_j\big).$$
Unwinding this shows exactly how $\phi_{i,j}$ transforms under the same representation V (up to the usual conventions on inverses). Concretely, one can check that this coincides with a direct sum of copies of V.

- **Right action.** For $h \in G$,
$$(\phi_{i,j} \cdot h)(g) \;=\; \phi_{i,j}(g\,h) \;=\; v_i^*\big((g\,h) \cdot v_j\big).$$
More explicitly, if $h \cdot v_j = \sum_\ell a_{\ell j}(h)\,v_\ell$, then
$$(\phi_{i,j} \cdot h)(g) \;=\; v_i^*\Big(g\,\big(\sum_\ell a_{\ell j}(h)\,v_\ell\big)\Big) \;=\; \sum_\ell a_{\ell j}(h)\;v_i^*\big(g \cdot v_\ell\big).$$
In terms of matrix indices, this precisely corresponds to taking the inverse-transpose (i.e. the dual) of the action by G on V.

Conclusion: the right action is via V^*. We've shown that the matrix coefficients lie in $V \otimes V^*$ as a $G \times G$-module; as this is irreducible as a $G \times G$-module they must span the V-isotypic component of $k[G]$ by a dimension count.

This completes the identification
$$V \otimes \mathrm{Hom}_G(V, k[G]) \;\;\cong\;\; V \otimes V^* \quad \text{as a } (G \times G)\text{-module}.$$

We will now work towards proving the following basic fact about representations of finite groups, which is one of the last remaining loose ends before we dive into representations of specific families of finite groups.

Theorem. Let G be a finite group and k an algebraically closed field of characteristic zero. Let V be an irrep of G over k. Then $\dim V | \#G$.

We will recall a bit of arithmetic before jumping into the proof.

Some recollections on integrality. Let R be a commutive ring. An element $x \in R$ is integral if it satisfies a monic polynomial with integer coefficients.

Prop. TFAE:

(1) x is integral
(2) The subring of R generated by x is finitely-generated as a $\mathbb{Z}$-module.
(3) The subring of R generated by x is contained in a finitely-generated $\mathbb{Z}$-submodule of R.

Proof. (1) $\implies$ (2) $\implies$ (3) is obvious (though please explain why :P). (3) $\implies$ (2) follows from the fact that $\mathbb{Z}$ is Noetherian (or from the classification of finitely-generated Abelian groups). (2) $\implies$ (1) also follows from Noetherianity—there exists some N such that $\{1, x, \cdots, x^N\}$ generate the given subring as an Abelian group, i.e. x^{N+1} is in their $\mathbb{Z}$-span, and hence satisfies a monic integer polynomial.

Cor. Integral elements of R form a subring.

Integrality of characters. Now again let G be a finite group and k an algebraically closed field of chracteristic zero. Let
$$\rho : G \to GL_n(k)$$
be a representation.

Prop. (1) The values of the character of ρ, $\chi_\rho(g)$, are algebraic integers. (2) Let $u = \sum_{g \in G} u(g)g$ be an element of $Z(k[G])$, the center of the group algebra. Suppose the elements $u(g) \in k$ are algebraic integers. Then u is integral over $\mathbb{Z}$.

Proof. (1) $\chi_\rho(g)$ is a sum of roots of unity, hence a sum of algebraic integers, hence an algebraic integer itself.

(2) As algebraic elements form a subring, it suffices to do this for the case where u is the indicator function of a conjugacy class. But the sub-$\mathbb{Z}$-module of $Z(k[G])$ generated by these indicator functions is in fact a subring. Thus each is contained in a finitely-generated $\mathbb{Z}$-module, and is hence algebraic.

Cor. Let ρ be an irrep of G, and let $u \in Z(k[G])$ be as above, i.e. a linear combination of group elements with coefficients algebraic integers. Then

$$\frac{1}{\dim \rho} \sum_{g \in G} u(g)\chi_\rho(g) \in k$$

is an algebraic integer.

Proof. We will define a ring homomorphism $Z(k[G]) \to k$ sending u to the element above; as u is integral, it maps to an integral element of k.

Namely, recall that $Z(k[G])$ acts on ρ by G-homomorphisms, i.e. that action induces a natural ring map

$$Z(k[G]) \to \mathrm{Hom}_G(\rho, \rho) = k.$$

In fact we already computed this map in the course of proving characters span class functions; it sends u to

$$\frac{|G|}{\dim \rho}\langle u, \chi_{\rho^\vee}\rangle,$$

which is the same as the claimed sum. (Recall that we proved this as follows: the action is by a scalar matrix, by Schur, so it suffices to compute its trace. This gives

$$\sum u(g)\chi_\rho(g) = |G|\langle u, \chi_{\rho^\vee}\rangle.$$

Dividing by the dimension gives the claim.)

We can now prove the:

Theorem. Let G be a finite group and k an algebraically closed field of characteristic zero. Let V be an irrep of G over k. Then $\dim V | \#G$.

Proof. Let ρ be an irrep and set $u = \sum_{g \in G} \chi_\rho(g^{-1})g$. By the above, we have

$$\frac{1}{\dim \rho} \sum_{g \in G} u(g)\chi_\rho(g) = \frac{|G|}{\dim \rho}\langle \chi_\rho, \chi_\rho\rangle = \frac{|G|}{\dim \rho} \dim \mathrm{Hom}_G(\rho, \rho) = \frac{|G|}{\dim \rho}.$$

Thus $\frac{|G|}{\dim \rho}$ is an algebraic integer; as it is also a rational number, it is in fact an integer as desired.

Representation theory of the symmetric group.

We'll first discuss some basic results, and then go into the proofs. I'm (loosely) following some mix of Fulton-Harris and Etingof's notes.

Partitions and Conjugacy Classes

Key Facts:

- The number of irreducible representations of the symmetric group S_n equals the number of its conjugacy classes.
- Conjugacy classes in S_n are in bijection with partitions of n.
- In fact, irreps of S_n are indexed by partitions of n, often represented by Young diagrams (Ferrers diagrams). This is an unusual situation—while the number of conjugacy classes is always the same as the number of representations, it's unusual that there is some natural bijection.

Young diagram associated to a partition. A partition of n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. The corresponding Young diagram has λ_i boxes in the ith row (from top to bottom).

Conjugate partition. Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, its *conjugate* partition λ' is obtained by reflecting the Young diagram along its main diagonal. Equivalently, λ'_j is the number of boxes in the jth column of the diagram for λ.

Projectors and Young Symmetrizers

We describe irreps of S_n in terms of certain idempotent elements (projections) in the group algebra $\mathbb{C}[S_n]$. We will apply these to the (left) regular representation to obtain each irreducible representation.

Number the boxes in a Young diagram from left to right and top to bottom, e.g.

1 2 3 4
5 6 7
8
9.

A labeled Young diagram is a *Young tableau.*

Row and column stabilizers

- Let $P \subseteq S_n$ be the subgroup of all permutations that preserve each row of a fixed Young diagram (i.e., permutations acting within each row).
- Let $Q \subseteq S_n$ be the subgroup of all permutations that preserve each column of the same Young diagram.

In the group algebra $\mathbb{C}[S_n]$, define

$$a = \sum_{p \in P} e_p, \quad b = \sum_{q \in Q} \operatorname{sgn}(q)\, e_q.$$

Here e_g denotes the group algebra element corresponding to $g \in S_n$, and $\operatorname{sgn}(q)$ is the sign of q.

Motivation: action on tensor powers

Let V be a vector space on which S_n acts by permuting tensor factors in $V^{\otimes n}$.

- The element a *symmetrizes* along the rows and hence projects onto

$$\underbrace{\mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_m}(V)}_{\text{row-based symmetrization}},$$

up to an isomorphism.
- The element b *alternates* along the columns and projects onto a tensor product of exterior powers indexed by the conjugate partition λ', i.e.,

$$\Lambda^{\lambda'_1}(V) \otimes \cdots \otimes \Lambda^{\lambda'_k}(V).$$

The Young symmetrizer. Set

$$c \;=\; a\,b.$$

This element is called the *Young symmetrizer* associated to the partition λ.

Example For a single column partition $(1, 1, \ldots, 1)$ (of length n), c gives the *sign representation.* For a single row partition (n), c gives the *trivial representation* (when applied to the regular representation).

Irreducibility and Idempotency

Theorem 1. *A suitable nonzero scalar multiple of $c = ab$ is an idempotent in $\mathbb{C}[S_n]$. Its image, when acting on the regular representation, is irreducible and is denoted by V_λ. Distinct partitions give non-isomorphic irreps, and every irrep arises for a unique partition.*

Corollary Every irreducible representation of S_n can be defined over $\mathbb{Q}$.

Examples.

- $\lambda = (n)$ yields the *trivial representation*, with c a scalar multiple of the Reynolds operator (averaging over all of S_n).
- $\lambda = (1, 1, \ldots, 1)$ yields the *sign representation.*
- $\lambda = (2, 1)$ for S_3 corresponds to the *standard representation* of S_3.
- For S_4, partitions (4), $(1, 1, 1, 1)$, $(3, 1)$, $(2, 1, 1)$, $(2, 2)$ give all irreps: trivial, sign, standard, standard $\otimes$ sign, and the remaining two-dimensional one pulled back from S_3.

Remark In general, partitions of the form $(d, 1, \ldots, 1)$ correspond to the various *exterior powers of the standard representation.*

Character theory. Frobenius gave explicit character formulas for V_λ; see Fulon Harris, IV for details. We will not prove those formulas here.

Dimension (Hook-Length) Formula

Proposition (Dimension Formula) Label each box b in a Young diagram by (boxes to the right of b) + (boxes below b) + 1.

So for example
6 4 3 1
4 2 1
1.

These are called the *hook lenghts*, and the dimension of the irrep V_λ is given by

$$\dim V_\lambda \;=\; \frac{n!}{\prod_{b\in \text{Young diagram of } \lambda} \text{hook\_length}(b)}.$$

Example For the standard representation of S_n, partition $\lambda = (n-1,1)$, one finds

$$\dim V_{(n-1,1)} \;=\; n!/n(n-2)(n-3)\cdots 2\cdot 1 = n-1.$$

as the hook lengths are

$$\begin{array}{ll} n & 1 \\ n-2 & \\ n-3 & \\ \vdots & \end{array}$$

Proofs of Key Lemmas, following Etingof

Lemma 2. *We have*

$$c = ab \;=\; \sum_{g\in PQ} \operatorname{sgn}(g_Q)\, e_g,$$

where $g = pq$ with $p \in P$ and $q \in Q$, and $\operatorname{sgn}(g_Q) = \operatorname{sgn}(q)$.

Proof. Direct computation. The point is that each element of S_n can appear at most once in this product, because $P \cap Q = \{e\}$.

Lemma 3. *Let $x \in \mathbb{C}[S_n]$ and λ a partition. Then*

$$a_\lambda\, x\, b_\lambda \;=\; \ell_\lambda(x)\, c_\lambda,$$

where ℓ_λ is a linear function.

Proof. If $g \in P_\lambda Q_\lambda$, then g has a unique representation $g = pq$ with $p \in P_\lambda$ and $q \in Q_\lambda$. In this case,

$$a_\lambda\, g\, b_\lambda \;=\; (-1)^q\, c_\lambda.$$

Therefore, to prove the required statement, we need to show that if g is not in $P_\lambda Q_\lambda$, then $a_\lambda\, g\, b_\lambda = 0$.

To see why, it suffices to find a transposition t such that $t \in P_\lambda$ and $g^{-1}tg \in Q_\lambda$. Then

$$a_\lambda\, g\, b_\lambda \;=\; a_\lambda\, t\, g\, b_\lambda \;=\; a_\lambda\, g\,(g^{-1}\, t\, g)\, b_\lambda \;=\; -\,a_\lambda\, g\, b_\lambda,$$

which forces $a_\lambda\, g\, b_\lambda = 0$.

Equivalently, we must find two elements i, j that lie in the same row of the tableau $T = T_\lambda$ and in the same column of the tableau $T' = gT$ (where gT is obtained by permuting the entries of T via g). If no such pair (i, j) exists, then we claim that $g \in P_\lambda Q_\lambda$; more precisely, there exist $p \in P_\lambda$ and $q' \in Q'_\lambda := g\, Q_\lambda\, g^{-1}$ so that $pT = q'\, T'$. From this, one concludes $g = p\, q'^{-1}$ and $q = g^{-1}\, q'\, g \in Q_\lambda$.

Indeed, any two elements in the first row of T must go to different columns of T'. Hence there exists $q'_1 \in Q'_\lambda$ that repositions all these first-row elements into the first row of T'. Then we can find $p_1 \in P_\lambda$ so that $p_1 T$ and $q_1' T'$ share the same first row. Repeating this procedure for the second row (finding p_2, q_2' so that $p_2 p_1 T$ and $q_2' q_1' T'$ match on the first two rows) and continuing row by row, we eventually construct the desired elements p, q'. The lemma is proved. $\square$

Lexicographic Ordering on Partitions. We say $\lambda > \mu$ if, in the first place where they differ, we have $\lambda_i - \mu_i > 0$.

Lemma 4. *If $\lambda > \mu$, then*

$$a_\lambda \, \mathbb{C}[S_n] \, b_\mu \; = \; 0.$$

Proof. Similarly to the previous lemma, it suffices to show that for any $g \in S_n$, there is a transposition $t \in P_\lambda$ such that $g^{-1} \, t \, g \in Q_\mu$. Let $T = T_\lambda$ and $T' = g \, T_\mu$. We claim that there exist two integers that lie in the same row of T but in the same column of T'. Indeed, if $\lambda_1 > \mu_1$, the pigeonhole principle shows this already in the first row.

Otherwise, if $\lambda_1 = \mu_1$, then by an argument analogous to the proof of the preceding lemma, we can find $p_1 \in P_\lambda$ and $q_1' \in g \, Q_\mu \, g^{-1}$ so that $p_1 \, T$ and $q_1' \, T'$ agree in the first row. We then repeat this process for the second row, and so on. After $i - 1$ such steps, we will have $\lambda_i > \mu_i$, which forces two entries of row i of the first tableau to occupy the same column in the second tableau. This completes the proof. $\square$

Lemma 5. *The element c_λ is proportional to an idempotent. Namely,*

$$c_\lambda^2 \; = \; \frac{n!}{\dim V_\lambda} \, c_\lambda.$$

Proof. By Lemma 3, c_λ^2 is already known to be proportional to c_λ. The claimed constant is computed by computing the trace of both sides; leave this to the students. $\square$

Lemma 6. *Let A be an algebra and e be an idempotent in A. Then for any left A-module M, one has*

$$Hom_A(Ae, M) \; \cong \; eM,$$

where an element $x \in eM$ corresponds to the map $f_x : Ae \to M$ given by $f_x(a) = a \, x$ for $a \in Ae$.

Proof. Note that $1 - e$ is also an idempotent in A. Hence the statement follows immediately from the isomorphism

$$\mathrm{Hom}_A(A, M) \; \cong \; M$$

and the decomposition

$$A \; = \; Ae \, \oplus \, A \, (1 - e).$$

$\square$

Proof of Theorem 1. Let $\lambda \geq \mu$. Then by the two lemmas above,

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) \;=\; \mathrm{Hom}_{S_n}\big(\mathbb{C}[S_n]\, c_\lambda,\; \mathbb{C}[S_n]\, c_\mu\big) \;=\; c_\lambda\, \mathbb{C}[S_n]\, c_\mu.$$

By Lemma 4, this space vanishes for $\lambda > \mu$, and by Lemmas 3 and 5 it is one-dimensional for $\lambda = \mu$. Hence each V_λ is irreducible, and $V_\lambda \ncong V_\mu$ if $\lambda \neq \mu$. Since the number of partitions of n equals the number of conjugacy classes in S_n, the representations V_λ exhaust all irreducible representations of S_n. The theorem is proved.