CLOSED SUBGROUPS OF LIE GROUPS ARE LIE SUBGROUPS

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We begin with a relatively easy theorem, which points in the direction of the proof.

Theorem 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $H \hookrightarrow G$ be a be a subgroup (in the category Grp). Then H is a Lie subgroup if and only if there exists a subspace $V \subset \mathfrak{g}$ and neighborhoods $0 \in U \subset \mathfrak{g}$ and $e \in W \subset G$ such that*

$$\exp|_{U\cap V} : U \cap V \to W \cap H$$

is a diffeomorphism.

Proof. If H is a Lie subgroup, the theorem is obvious. For the converse, note that for $h \in H$, we may consider $\exp|_{U\cap V}^{-1}(h^{-1} \cdot -)$, which is a map $hW \cap H \to U \cap V$. It is clear that these maps cover H and are mutually compatible (as they are diffeomorphisms onto their image in G) and so they induce a manifold structure on H; by construction, this manifold structure is compatible with that of G. $\square$

Theorem 2. *Let G be a Lie group, and $H \hookrightarrow G$ a subgroup (in the category Grp), with closed image. Then H is an embedded Lie subgroup.*

Let $\mathfrak{h}$ be the subset of $\mathfrak{g}$ defined as

$$\mathfrak{h} := \{x \in \mathfrak{g} \mid \exp(tx) \in H, \forall t \in \mathbb{R}\}.$$

That is, the $x \in \mathfrak{g}$ such that the one-parameter subgroup generated by x is entirely contained in H.

Lemma 1. *$\mathfrak{h}$ is a subspace of $\mathfrak{g}$.*

Before proving this we need an auxiliary lemma:

Lemma 2. *Let $x, y \in \mathfrak{g}$. Then*

$$\lim_{n\to\infty} \left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n = \exp(t(x + y)).$$

Proof of Lemma 2. Consider the path given by $\gamma : \mathbb{R} \to G$, where $\gamma(t) = \exp(tx)\exp(ty)$. Let $0 \in U \subset \mathfrak{g}$, $e \in W \subset G$ be neighborhoods such that $\exp|_U : U \to W$ is a diffeomorphism, and let $Z(t) = \exp|_U^{-1} \circ \gamma(t)|_{U'}$, where $0 \in U' \subset \mathbb{R}$ is a sufficiently small neighborhood of zero. Then we have

$$\gamma(t) = \exp(Z(t))$$

for $t \in U'$. Note that $dZ|_0 = d\exp^{-1} \circ d(\exp(tx)\exp(ty))|_0 = d(\exp(tx)\exp(ty))|_0 = x + y$ so taking Taylor series, we have

$$\exp(tx)\exp(ty) = \exp(t(x + y) + O(t^2))$$

on some small neighborhood of $0 \in \mathbb{R}$.

But then

$$\begin{aligned}\lim_{n\to\infty} \left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n &= \lim_{n\to\infty} \exp\left(\frac{t}{n} \cdot (x + y) + O((t/n)^2)\right)^n \\ &= \lim_{n\to\infty} \exp(t \cdot (x + y) + O(t^2/n)) \\ &= \exp(t(x + y))\end{aligned}$$

as desired. $\square$

Now we can prove that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$.

Proof of Lemma 1. It is clear that $\mathfrak{h}$ is closed under scalar multiplication. To see that it is closed under addition, note that if $x, y \in \mathfrak{h}$, then $\left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n$ is in H; as H is closed, we thus have that

$$\lim_{n\to\infty} \left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n = \exp(t(x+y)) \in H$$

by Lemma 2. □

So to complete the proof of the theorem, we want to find open sets $0 \in U \subset \mathfrak{g}, e \in W \subset G$ such that $\exp|_{U\cap V} : U \cap V \to W \cap H$ is a diffeomorphism. Let $\mathfrak{f} \subset \mathfrak{g}$ be a complementary subspace to $\mathfrak{h}$. Let $\alpha : \mathfrak{h} \times \mathfrak{f} \to \mathfrak{g}$ be given by $\alpha(x, y) = \exp(x)\exp(y)$. Note that $d\alpha|_0 = \mathrm{id}_{\mathfrak{g}}$, so α is a diffeomorphism when restricted to a neighborhood $U_{\mathfrak{h}} \times U_{\mathfrak{f}} \subset \mathfrak{h} \times \mathfrak{f}$ of 0.

Lemma 3. *There exists an open* $0 \in U_{\mathfrak{f}} \subset \mathfrak{f}$ *such that*

$$H \cap \exp(U_{\mathfrak{f}} - \{0\}) = \emptyset.$$

Proof of Lemma 3. Assume the contrary. Then there exists a sequence $(x_j) \in U_{\mathfrak{f}}$ with $\exp(x_j) \in H$, $x_j \to 0$. Let $||\cdot||_{\mathfrak{f}}$ be a norm on $\mathfrak{f}$ and let $(x'_j) = (x_j/||x_j||_{\mathfrak{f}})$; by the compactness of the unit ball, there is some subsequence (x'_{j_k}) converging to $y \in \mathfrak{f}$ with $y \neq 0$. Let $t_{j_k} = ||x_{j_k}||_{\mathfrak{f}}$. As $t_{j_k} \to 0$, we may choose integers $n_{j_k}(t)$ such that $t_{j_k} n_{j_k}(t) \to t$. But then we have

$$\exp(ty) = \exp(t \cdot \lim x'_{j_k}) = \exp(\lim n_{j_k}(t) t_{j_k} x'_{j_k}) = \lim \exp(n_{j_k}(t) x_{j_k}) = \lim \exp(x_{j_k})^{n_{j_k}(t)} \in H$$

But then $y \in \mathfrak{h}$, so $y = 0$, which is a contradiction, as $||y||_{\mathfrak{f}} = 1$. □

So we may choose $U_{\mathfrak{f}}$ as in Lemma 3 such that $\alpha : U_{\mathfrak{f}} \times U_{\mathfrak{h}} \to G$ is a diffeomorphism on its image.

Now we claim that $\exp(U_{\mathfrak{h}} \times \{0\})$ contains some neighborhood of the identity in H. Indeed, let $S = \mathrm{Im}(\alpha|_{U_{\mathfrak{h}} \times U_{\mathfrak{f}}})$; $S \subset G$ is an open neighborhood of e. For $x \in S \cap H$ we have that $x = \exp(x')\exp(y')$ for $x' \in U_{\mathfrak{h}}, y' \in U_{\mathfrak{f}}$; but then $x, \exp(x') \in H$ and so $\exp(y') \in H \cap \exp(U_{\mathfrak{f}})$, so by Lemma 3 $y' = 0$. Thus x is in the image of exp.

So we may take $U_{\mathfrak{h}} \times U_{\mathfrak{f}}$ to be our U from Theorem 1, and take S to be our W, completing the proof.