Yet Another Proof of the Fundamental Theorem of Algebra

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1 Strategy

We prove the fundamental theorem of algebra, using only elementary techniques from calculus, point-set topology, and linear algebra; this proof apparently does not appear in the extensive literature on the subject [1], [2]. The exposition is essentially self-contained.

Theorem 1 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial with complex coefficients has a root in* $\mathbb{C}$.

This is the strategy of the proof. Let $X_n \simeq \mathbb{C}^n$ be the space of degree n monic polynomials with complex coefficients, via the identification $(a_1, ..., a_n) \mapsto z^n + \sum_{i=1}^n a_i z^i$. Let $D \subset X_n$ be the zero locus of the discriminant, and let $R \subset X_n \setminus D$ be the set of polynomials with non-zero discriminant which have at least one complex root. We show:

1. $X_n \setminus D$, the set of monic degree n polynomials with non-zero discriminant, is connected.

2. R, the set of monic degree n polynomials with non-zero discriminant which have at least one root, is both open and closed in $X_n \setminus D$. As R is nonempty it is thus equal to $X_n \setminus D$, so every monic degree n polynomial with non-zero discriminant has a root.

3. By induction on n, every polynomial with zero discriminant has a root.

2 Preliminaries

The following preliminary lemma is the only part of the argument that uses that the ground field is $\mathbb{C}$, rather than $\mathbb{R}$.

Lemma 1. *Let* $V \subset \mathbb{C}^n$ *be the zero locus of some polynomial* $p(x) = p(x_1, ..., x_n)$*. Then* $\mathbb{C}^n \setminus V$ *is path-connected, and thus connected.*

Proof. Let $y, z \in \mathbb{C}^n \setminus V$ be two points in the complement of V. Consider the set $S = \{cy + (1-c)z \mid c \in \mathbb{C}\} \subset \mathbb{C}^n$, which is a complex line connecting y and z. Then $S \cap V$ is a finite set, as $p(cy+(1-c)z)$ is a polynomial in the single complex variable c, and thus has at most finitely many zeros. In particular, $S \setminus (S \cap V)$ is homeomorphic to the complex plane with finitely many points removed, and so is path connected. Thus there is a path in $S \setminus (S \cap V)$ connecting y and z. □

We will also need an easy lemma bounding the size of the roots of a monic polynomial in terms of its coefficients.

Lemma 2. *Let $\{f_\alpha\}$ be a set of monic degree n polynomials whose coefficients all lie in some bounded region of $\mathbb{C}$. Then there exists $C > 0$ such that if z is a zero of f_α for some α, then $|z| < C$.*

Proof. This is immediate from the fact that

$$\frac{f_\alpha(z)}{z^n} \to 1 \text{ as } |z| \to \infty$$

uniformly in α. □

Finally, we introduce the resultant and discriminant. Let k be a field and let $f, g \in k[x]$ be non-constant polynomials with coefficients in k. Then there is a map

$$\psi_{f,g} : k[x]/(f) \oplus k[x]/(g) \to k[x]/(fg)$$

given by

$$(a + (f), b + (g)) \mapsto ag + bf + (fg).$$

By the chinese remainder theorem, this map is a k-vector space isomorphism if and only if $\gcd(f, g) = 1$. Define the resultant

$$R_{f,g} = \det(\psi_{f,g}).$$

Note that by the previous remark, $R_{f,g} = 0$ if and only if f, g have a common factor. Taking $k = \mathbb{C}(a_0, a_1, ..., a_n, b_0, b_1, ..., b_m)$ with

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,\ g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

and choosing bases for $k[x]/(f), k[x]/(g), k[x]/(fg)$ gives a formula for $R_{f,g}$ as a polynomial in the coefficients of f, g for general polynomials f, g with complex coefficients.

Now let f be any polynomial of degree at least 2 with complex coefficients, and define the discriminant $D_f = R_{f,f'}$, where f' is the derivative of f. Note that D_f is a polynomial in the coefficients of f. Furthermore, $D_f = 0$ if and only if f has a factor in common with its derivative.

3 The Proof

We now prove the fundamental theorem of algebra (Theorem 1).

Let $X_n \simeq \mathbb{C}^n$ be the space of degree n monic polynomials with complex coefficients, via the identification $(a_1, ..., a_n) \mapsto z^n + \sum_{i=1}^n a_i z^i$; we endow X_n with the analytic topology. Let $D \subset X_n, D := \{f \in X_n \mid D_f = 0\}$ be the set of polynomials f with discriminant 0. Namely, D consists of those polynomials which have a factor in common with their derivative. Note that D is a closed subset of X_n, as it is the zero set of a polynomial. Define $R \subset X_n \setminus D$, by

$$R = \{f \in X_n \setminus D \mid \exists z \in \mathbb{C} \text{ such that } f(z) = 0\}.$$

That is, R consists of those polynomials, with non-zero discriminant, which have a root in $\mathbb{C}$. Note that R is non-empty; for example, it contains $z^n - 1$.

We claim that R is open in $X_n \setminus D$, in the subspace topology. To see this, let $\text{ev} : \mathbb{C} \times (X_n \setminus D) \to \mathbb{C}$ be the evaluation map $(z, p) \mapsto p(z)$. Consider $f \in R$; by definition, f has a root t, so $\text{ev}(t, f) = 0$. Furthermore, $(\frac{d}{dt}\text{ev})(t, f) = f'(t)$ is non-zero, as otherwise $(z - t)$ divides both f and f', and thus $D_f = 0$, which contradicts the fact that $f \notin D$.

Thus, by the implicit function theorem, there exists an open neighborhood $U \subset X_n \setminus D$ with $f \in U$, and a function $r : U \to \mathbb{C}$ such that $r(f) = t$ and $g(r(g)) = \text{ev}(r(g), g) = 0$ for all $g \in U$. That is, we have found a neighborhood U of f and a function on U parametrizing roots of polynomials in U; in particular, all of the polynomials in U have a root. Thus $U \subset R$, and so R is open.

Now, we claim R is closed in $X_n \setminus D$. Let $f_k \to f$ in $X_n \setminus D$, with $f_k \in R$ for all k; we wish to show that f has a root in $\mathbb{C}$. As each $f_k \in R$, there exists $z_k \in \mathbb{C}$ with $f_k(z_k) = 0$. By Lemma 2, the z_k are bounded, and so there exists a convergent subsequence $z_{\alpha_k} \to z$. So replacing $\{f_j\}, \{z_j\}$ by subsequences, we may assume $z_j \to z$. We claim $f(z) = 0$, and thus $f \in R$. Indeed, we have

$$|f(z) - f_k(z_j)| \leq |f(z) - f(z_j)| + |f(z_j) - f_k(z_j)|. \qquad (*)$$

Taking j, k large, we may make the right hand side of $(*)$ arbitrarily small, by the continuity of f and the fact that the f_k converge to f pointwise. Now taking $j = k$ large, $f_k(z_j) = 0$, so we may make $|f(z)|$ arbitrarily small. Thus $f(z) = 0$ as desired.

So R is both open and closed in $X_n \setminus D$. But by Lemma 1, $X_n \setminus D$ is connected, so $R = X_n \setminus D$. In particular, every polynomial of degree n with non-zero discriminant has a root.

It remains only to show that those degree n polynomials f with zero discriminant have a root. But such polynomials f have a factor g in common with their derivatives f'. The degree of g is less than that of f, and so we are done by induction on n, as the degree 1 case is trivial.

References

[1] B. Fine and G. Rosenberger. *The Fundamental Theorem of Algebra*. Undergraduate Texts in Mathematics. Springer Verlag, Berlin, 1997.

[2] Anweshi (mathoverflow.net/users/2938). Ways to prove the fundamental theorem of algebra. MathOverflow. URL: http://mathoverflow.net/questions/10535 (version: 2010-01-04).