MAT138H1 HW2, DUE 10/12/2022

(1) Show that there are infinitely many primes p such that the remainder upon dividing p by 4 is 3. That is, show that there are infinitely many primes p of the form $4k+3$. Hint: Assume not, and let $3, p_1, p_2, \cdots, p_N$ be the set of primes of the form $4k+3$. Now consider the expression $4p_1p_2\cdots p_N+3$.

(2) (JDH 3.3) Show that if a, b are positive integers such that d is the greatest common divisor of a and b, then there exist integers x, y such that

$$d = ax + by.$$

Show that d is the smallest positive integer with this property.

(3) (JDH 3.6) Show that if p, q are primes such that p divides q, then $p = q$.

(4) (JDH 3.7) Show that if $p > 3$ is prime, then the remainder when one divides p by 6 is either 1 or 5. That is, every prime $p > 3$ is of the form $6k+1$ or $6k+5$.

(5) (JDH 3.8) Let a, b, c be positive integers such that abc is a multiple of 6. Show that at least one of ab, bc, ac is a multiple of 6.

(6) Recall that in Peano arithmetic, addition is defined by the properties

$$a + 0 = a, a + S(b) = S(a + b).$$

Show that for all a, $0 + a = a$. (Hint: use induction on a.) Justify ALL your steps using the axioms. Note that commutativity of addition is not one of the axioms!

(7) Show, using the axioms of Peano arithmetic, that addition is commutative, i.e. $a + b = b + a$ for all a, b. (Hint: use induction on b.) Justify ALL your steps using the axioms.

(8) In Peano arithmetic, multiplication is defined by the properties

$$a \cdot 0 = 0, a \cdot S(b) = a \cdot b + a.$$

Recall that 1 is defined to be $S(0)$. Show that $a \cdot 1 = a$ for all a. Do you need to use induction?

(9) Show that $0 \cdot b = 0$ for all b. (Hint: use induction on b.) Justify ALL your steps using the axioms.

(10) Show, using the axioms of Peano arithmetic, that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

for all a, b, c. Justify all your steps.

(11) (JDH 4.1) Show, using induction, that $2^n < n!$ for all $n \geq 4$.

(12) (JDH 4.3) The Fibonacci numbers f_n are defined by $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n > 1$. Show, using induction, that $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$.

(13) Show, using induction, that

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(14) (JDH 4.13) True or false: every flying polka-dotted elephant in this room is smoking a cigar. Justify your answer.

(15) Try out the Natural Numbers Game (`https://www.ma.imperial.ac.uk/~buzzard/xena/natural_number_game/`). Try to get to the end of multiplication world (it's OK if you get stuck). Write a few sentences about your experience, and your thoughts on computer-aided proof. (You will automaticallly get full credit on this question as long as you write something reasonable, I just want to hear your thoughts.)