## MAT138H1 HW3, DUE 10/19/2022

(1) Define the sequence $B_n$ by $B_0 = 0, B_1 = 1$, and $B_n = B_{n-1} + 2B_{n-2}$ for $n \geq 2$. Find a simple formula for $B_n$ and prove it is true, using induction.

(2) (JDH 7.1) True or false: in the game Twenty-One, making a mistake early on does not matter so much, since you can recover by the end by adopting the winning strategy. [Make a precise mathematical claim about this and prove it.]

(3) (JDH 7.16) Describe a natural 3-dimensional version of tic-tac-toe, played on a $3 \times 3 \times 3$ grid, and prove that the first player has a winning strategy.

(4) Recall the game Binary Biscuits$^{\text{TM}}$ from class. Two players start with $n$ biscuits. The players alternate, each eating exactly $2^m$ biscuits on their turn, where $m$ is some non-negative integer they get to choose on each turn. (So for example, the first player could eat 1 biscuit, then the second could eat 8 biscuits, and then the first could eat 1024 biscuits, and then the second could eat 1 biscuit, and so on.) The player who eats the last biscuit wins. For which $n$ does the first player have a winning strategy? Describe the winning strategy and prove your answer is correct.

(5) The game Very Polite Binary Biscuits$^{\text{TM}}$ is played just like Binary Biscuits, except the player to eat the last biscuit loses. For which $n$ does the first player have a winning strategy? Prove your answer is correct.

(6) Trinary Triscuits$^{\text{TM}}$ is just like Binary Biscuits$^{\text{TM}}$, (played with triscuits instead of biscuits) except that on their turn, a player must eat exactly $3^m$ triscuits for some non-negative integer $m$. If the players start with $n$ biscuits, for which $n$ does the first player have a winning strategy? Describe the winning strategy and prove your answer is correct.

(7) In the game I Drink Your Milkshake$^{\text{TM}}$, there are two players, $A$ and $B$, who have $n > 0$ milkshakes between them. The players are also given a collection of positive integers $\{1, k_1, \cdots, k_r\}$. Player $A$ and $B$ take turns, with each drinking exactly $1, k_1, k_2, \cdots, k_{r-1}$ or $k_r$ milkshakes. *Except* Player $A$ goes first, and on their very first turn they can choose to drink 1 additional milkshake (so, on their very first turn, Player $A$ can drink exactly $1, 2, k_1, k_1 + 1, k_2, k_2 + 1, \cdots, k_{r-1}, k_{r-1} + 1, k_r$ or $k_r + 1$ milkshakes). After that first turn, the players have to follow the same rules, i.e. Player A never again

gets this special privilege. The player who drinks the last milkshake wins.

Prove that no matter what $n, k_1, \cdots, k_r$ are, Player $A$ has a winning strategy. (Hint: use a strategy-stealing argument.)

*Challenge (not for credit):* Can you find an explicit winning strategy for Player A?