MAT138H1 HW4, DUE 10/31/2022

(1) (JDH 5.14) Consider a version of the Escape! game which is played on an $n \times n$ board, modifying the rules so that a stone that would have been placed outside this region is simply not placed. Show that every sequence of legal moves leads to a completely empty board. (There is a hint in the book.)

(2) Let p be prime. Show that for any integer k with $0 < k < p$, p divides $\binom{p}{k}$. Is the analogous statement true if p is not prime?

(3) Use problem (2) and the fact that

$$(1+x)^p = \sum_{k=0}^{p} \binom{p}{k} x^k$$

to show that if n is an integer and p is prime, then the remainder of n^p when divided by p is the same as the remainder of n when divided by p. Hint: use induction on n.

Challenge (not for credit): can you find a different proof of this theorem by counting ways of arranging p beads, each of which could have one of n different colors, in a circle?

(4) (JDH 5.9) Prove or refute the following statement: if you place a chessboard pattern on an $n \times m$ grid, there will always be an equal number of light and dark squares.

(5) Let $n > 5$ be an integer. Suppose you are given $n + 5$ integers between 1 and $2n$, say $1 \leq x_1, \cdots, x_{n+5} \leq 2n$. Show that there exist distinct i, j, k, l such that

$$|x_i - x_j| + |x_k - x_l| \leq 2.$$

Hint: use the pigeonhole principle.

(6) Consider a standard deck of 52 cards, with 26 red cards and 26 blue cards. Alice and Bob each draw a card; if both cards are red, Alice gets a point, and if both cards are black Bob gets a point; in either case the cards are destroyed. If one card is red and one is black, the cards are destroyed and neither Alice nor Bob gets a point. The game is over when all the cards are gone. Out of the 52! ways of shuffling the deck, how many end in a tie? Prove your answer is correct.