LINEAR INDEPENDENCE OVER $\mathbb{Q}$ AND TOPOLOGY

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The idea of this note is to relate the linear independence (or dependence) of sets of complex numbers to the topological properties of certain spaces.

By the *Riemann surface of a holomorphic function* h on some region of U, I mean the connected component of the étale space of the sheaf of holomorphic functions containing the germs of h. One may alternatively construct this surface as follows. Let M_h be the set of germs of all analytic continuations of h, and given h defined on $B_r(x)$, let the set of germs of h on $B_r(x)$ be in a basis of open sets. This has an obvious complex analytic structure, via the map identifying the germs of h over $B_r(x)$ with $B_r(x)$ itself. There are two natural holomorphic maps from $M_h \to \mathbb{C}$. The first, $\pi_h : M_h \to \mathbb{C}$ sends a germ of h at x to x, namely, $[h]_x \mapsto x$. The latter, $e_h : M_h \to \mathbb{C}$, is given by evaluation. Namely, $e_h : [h]_x \mapsto [h]_x(x)$.

If h is a multivalued function, it's worth thinking of this space as the graph of the relation h; one may embed $h(z)$ into $\mathbb{C}^2$ via (π_h, e_h).

Now consider the following situation.

Let f be a meromorphic function on $\mathbb{C}$ with poles only of order 1, say at $z_1, z_2, \cdots, z_n$, and residues $\lambda_i = \mathrm{Res}_{z_i}(f)$. That is, f is a section of the sheaf of holomorphic functions $\mathcal{H}$ over $U = \mathbb{C} - \{z_1, \cdots, z_n\}$, and in particular, $M_f = \mathbb{C} - \{z_1, \cdots, z_n\}$. Let g be an antiderivative of f; that is, $g' = f$. Note that g is multivalued. There is a map $\pi_d : M_g \to M_f$ given by differentiation, which is an endomorphism of $\mathcal{H}$ and thus an endomorphism of its étale space, sending M_g to M_f. As this is a restriction of an endomorphism of the étale space, it is clearly a covering map. (Alternately, one may see that this is a covering map by noting that basis sets of M_g in our second construction are mapped isomorphically to basis sets of M_f.)

Let $V \subset \mathbb{C}$ be the $\mathbb{Q}$-vector space spanned by the λ_i, that is $V = \sum_i \mathbb{Q}\lambda_i$. Then I claim that the following sequence is exact:

$$H_1(M_g, \mathbb{Q}) \xrightarrow{H_1(\pi_d)} H_1(M_f, \mathbb{Q}) \xrightarrow{\int} V \longrightarrow 0$$

where $\int$ sends $[\gamma] \mapsto \int_{[\gamma]} f\, dz$.

Proof. (1) $\int$ is surjective: $H_1(M_f, \mathbb{Q}) \simeq \mathbb{Q}^n$, and has a basis given by fundamental cycles of M_f. Let $[v_i]$ be the fundamental cycle corresponding to the puncture at z_i; then $\int([v_i]) = \lambda_i$. But the λ_i span V, so the map is surjective.

(2) $\int \circ H_1(\pi_d) = 0$: Consider $[v] \in H_1(M_g, \mathbb{Q})$. Then we have

$$\begin{aligned}
\int \circ H_1(\pi_d) \circ [v] &= \int_{\pi_d^*([v])} f\ dz \\
&= \int_{[v]} \pi_d^*(f\ dz) \\
&= \int_{[v]} d(\pi_d^*(f)) \\
&= \int_{d([v])} \pi_d^*(f) \\
&= 0
\end{aligned}$$

as $d([v]) = 0$.

(3) $\ker(\int) \subset \operatorname{im}(H_1(\pi_d))$: By the Hurewicz theorem we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M_g) \arrow[r, hook, "\pi_1(\pi_d)"] \arrow[d, "h\otimes\mathbb{Q}"] & \pi_1(M_f) \arrow[d, "h\otimes\mathbb{Q}"] & & \\
H_1(M_g, \mathbb{Q}) \arrow[r, "H_1(\pi_d)"] & H_1(M_f, \mathbb{Q}) \arrow[r, two heads, "\int"] & V \arrow[r] & 0
\end{tikzcd}$$

where tensoring with $\mathbb{Q}$ makes sense as the h maps are abelianization. Consider some linear combination $\sum q_i[v_i]$, and multiply by N so that the coefficients are in $\mathbb{Z}$; we can then lift to $\pi_1(M_f)$. By this and the diagram, it suffices to show that $\operatorname{im}(\pi_1(\pi_d)) \supset \ker(\int \circ (h \otimes \mathbb{Q}))$. Consider a loop $[\ell] \in \pi_1(M_f)$ that maps to zero in V, e.g. choosing a representative, we have $\int_{S^1} \ell^*(f\ dz) = 0$. Then we claim it lifts to a loop in M_g. Indeed, let δ_t be such that $f|_{B_{\delta_t}(\ell(t))}$ has no singularities. Then letting $g_t : B_{\delta_t}(\ell(t)) \to \mathbb{C}$ be the map

$$z \mapsto \int_{\ell|_{[0,e^{2\pi i t}]}} f\ dz + \int_{\ell(t)}^{z} f\ dz,$$

we have that $\ell' : t \mapsto [g_t]_{\ell(t)}$ defines a loop in M_g (viewed as a space of germs), which is a lift of ℓ. (Note that the integral is well-defined by the choice of δ_t.

$\square$

So why do we care? Well, the kernel of $\int$ is precisely the vector space of relations over $\mathbb{Q}$ between the λ_i—to see this, note that as e.g. in (1) above, the map $\int$ acts on the basis $\{[v_i]\}$ of $H_1(M_f, \mathbb{Q})$ as $[v_i] \mapsto \lambda_i$. So the image of $H_1(\pi_d)$ is the vector space of relations, and this covering map gives a topological description of the linear dependence of the λ_i over $\mathbb{Q}$.

Now let $C_0 = \pi_1(M_f)$ and let $C_i = [C_{i-1}, C_{i-1}]$. Then C_i/C_{i+1} is abelian for all $i \geq 0$; we claim the sequence

$$\cdots \to C_{i+1}/C_{i+2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to C_i/C_{i+1} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to C_{i-1}/C_i \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \cdots \to C_0/C_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to H_1(M_g, \mathbb{Q}) \to H_1(M_f, \mathbb{Q}) \to V \to 0$$

is exact. The proof again follows from Hurewicz.