MOTIVIC ANALYTIC NUMBER THEORY

DANIEL LITT

1. Introduction

[I'd like to talk about some connections between topology, number theory, and algebraic geometry, arising from the study of configurations of points in a space.]

Topology: If X is a finite CW complex, we let $\mathrm{Sym}^n(X) := X^n/\Sigma_n$ (this is the configuration space of n *unordered* points on X). What is the homotopy type of $\mathrm{Sym}^n(X)$?

Number Theory: The Weil Conjectures: Let $X/\mathbb{F}_q$ be a variety. What does the generating function

$$\zeta_X(t) := \sum_{n=0}^{\infty} \#|\mathrm{Sym}^n(X)(\mathbb{F}_q)|t^n$$

look like? Note:

$$\frac{d}{dt}\log \zeta_X(t) = \sum_{n=0}^{\infty} \#|X(\mathbb{F}_{q^n})|t^n$$

(exercise.)

Analogous questions for zeta and L-functions associated to number fields...

Algebraic Geometry: Let X be a variety over a field k. What is the geometry of $\mathrm{Sym}^n(X)$? [E.g. what is the theory of 0-cycles on X?]

I'll discuss two types of answers to these questions, and their interplay.

(1) Stabilization: What does
$$\mathrm{Sym}^\infty(X) := \varinjlim_n \mathrm{Sym}^n(X)$$
look like? [Maps $\mathrm{Sym}^n(X) \to \mathrm{Sym}^{n+1}(X)$ induced by choice of basepoint.]

(2) Rationality: What is the relationship between $\mathrm{Sym}^n(X)$ and $\mathrm{Sym}^m(X)$? [I call this rationality because it relates to the rationality of certain generating functions.]

Topology: To answer (1), we have

Theorem 1 (Dold-Thom)**.** *Suppose X is a finite, connected CW complex. Then*

$$\mathrm{Sym}^\infty(X) \simeq_w \prod_{i\geq 1} K(H_i(X,\mathbb{Z}), i).$$

[The $K(G,n)$ are Eilenberg-Maclane spaces. I'll come back to this theorem later.]

There are many ways to answer (2), but here's a simple one. Namely, describe the generating function

$$Z_X(t) := \sum_{n=0}^{\infty} \chi_c(\mathrm{Sym}^n(X))t^n.$$

[Here χ_c denote the Euler characteristic with compact support.]

Two facts suggest an approach:

(1) Suppose $Z \subset X$ is closed. Then $\chi_c(X) = \chi_c(Z) + \chi_c(X \setminus Z)$.
(2) If $X = X_1 \sqcup X_2$, then $\mathrm{Sym}^n(X) = \coprod_{p+q=n} \mathrm{Sym}^p(X_1) \times \mathrm{Sym}^q(X_2)$.

Putting these facts together, we have

$$Z_X(t) = Z_{X_1}(t) \cdot Z_{X_2}(t).$$

[Let's package this information in a slightly different way, which will motivate our algebro-geometric approach later.]

Definition 2. Let

$$K_0(CW) = \left(\bigoplus_{X \text{ reasonable}} \mathbb{Z}[X] \right) / \{[X] = [Z] + [X \setminus Z] \mid Z \overset{cl}{\hookrightarrow} X\}$$

[here "reasonable" means, say, that X admits a stratification by finitely many copies of $\mathbb{R}^k$, e.g. compact manifolds, finite CW complexes, etc.]

Define multiplication by $[X] \cdot [Y] = [X \times Y]$.

Then (1) says that $\chi_c : K_0(CW) \to \mathbb{Z}$ is a homomorphism, and (2) says that

$$Z^{mot} : K_0(CW) \to 1 + tK_0(CW)[[t]],$$

$$Z_X^{mot}(t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)] t^n$$

is a homomorphism, with $Z_X(t) = \chi_c(Z_X^{mot}(t))$.

[There's actually a ring structure on the right-hand-side making this a ring homomorphism.]

Proposition 3. *$\chi_c : K_0(CW) \to \mathbb{Z}$ is an isomorphism.*

Proof. Clearly surjective.

$K_0(CW)$ is generated as an abelian group by $[\mathbb{R}^k] = [\mathbb{R}]^k$. So let's compute $[\mathbb{R}]$:

$$\mathbb{R} = \xleftarrow{\mathbb{R}} \bullet \xrightarrow{\mathbb{R}}$$

So $[\mathbb{R}] = 2[\mathbb{R}] + 1$, thus $[\mathbb{R}] = -1$. Thus χ_c is injective. □

Proposition 4. *$Z_X(t)$ is a rational function.*

Proof. Suffices to show $Z_X^{mot}(t)$ is a rational function. As $K_0(CW)$ is generated by [pt], suffices to show $Z_{\mathrm{pt}}^{mot}(t)$ is rational. But

$$Z_{\mathrm{pt}}^{mot}(t) = 1 + [\mathrm{pt}]t + [\mathrm{pt}]t^2 + \cdots = \frac{1}{1 - [\mathrm{pt}]t}.$$

Unwinding the isomorphism of Proposition 3, we have

$$Z_X(t) = (1-t)^{-\chi_c(X)}.$$

□

2. Some Motivic Questions

Algebraic Geometry: [This approach inspires the following definition:]

Definition 5. Let k be a a field. Let

$$K_0(\mathrm{Var}_k) = \left(\bigoplus_{X/k \text{ a variety}} \mathbb{Z}[X] \right) / \{[X] = [Z] + [X \setminus Z] \mid Z \overset{cl}{\hookrightarrow} X\}.$$

Let $[X] \cdot [Y] := [X \times Y]$.

[This ring is sometimes called "the ring of motives" or the Grothendieck ring of varieties.]

Some facts about this ring:

- Not a domain (Poonen, Kollar, etc.); not Noetherian...
- Homomorphisms:
  - $k = \mathbb{F}_q$: $[X] \mapsto \#|X(\mathbb{F}_q)|$
  - Any k: Poincaré polynomial, Euler characteristic in Grothendieck ring of Galois reps
  - $k = \mathbb{C}$: Hodge polynomial...
  - $k = \mathbb{C}$: $K_0(\mathrm{Var}_{\mathbb{C}})/\mathbb{L} \simeq \mathbb{Z}[SB]$ (Larsen, Lunts), where SB is the monoid of stable birational equivalence classes of smooth projective varieties.
- Open Questions
  - Is $\mathbb{L} := [\mathbb{A}^1]$ a zero divisor?

– If $[X] = [Y]$, are X and Y equidecomposable?

Our two questions become:

(1) Motivic Stabilization of Symmetric Powers (MSSP, Ravi Vakil, Melanie Wood): Does
$$\lim_{n\to\infty}[\mathrm{Sym}^n(X)]$$
converge in $\widehat{K_0(\mathrm{Var}_k)}$? [I'll discuss this question in two completions of the Grothendieck ring. Either complete $K_0(\mathrm{Var}_k)$ at $\mathbb{L}$ or invert $\mathbb{L}$ and complete at the dimension filtration.]

(2) (Motivic Weil Conjectures, Kapranov) Let
$$Z_X^{mot}(t) := \sum_{n=0}^{\infty}[\mathrm{Sym}^n(X)]t^n.$$
Is Z_X^{mot} rational? Does it satisfy a functional equation? [What is the analogue of the Riemann Hypothesis? Note that $Z_X^{mot}(t)$ specializes to $\zeta_X(t)$ under the map $[X] \mapsto \#|X(\mathbb{F}_q)|$.]

Theorem 6 (Disappointing theorem, Larsen and Lunts, 2002)**.** *Let X be a surface over $\mathbb{C}$. Then $Z_X^{mot}(t)$ is rational if and only if X is rational or birationally ruled.*

[Regardless, I want to discuss some ways that Z_X^{mot} behaves like a rational or meromorphic function. Let's do some examples:]

Example 1 ($\mathbb{L} := [\mathbb{A}^1]$). I claim that $\mathrm{Sym}^n(\mathbb{A}^1) \simeq \mathbb{A}^n$. "Proof": send an unordered tuple of elements of $\mathbb{A}^1$ to the coefficients of the monic polynomial with that tuple as its roots. [Actual proof—the fundamental theorem on symmetric polynomials.]

Example 2 ($\mathbb{L}[X]$). I claim $[\mathrm{Sym}^n(\mathbb{A}^1 \times X)] = \mathbb{L}^n[\mathrm{Sym}^n(X)]$. "Proof" (Totaro): This is surprisingly difficult, and uses Hilbert's Theorem 90 in its full generality.

Corollary 7.
$$Z_{\mathbb{L}}^{mot}(t) = \frac{1}{1-\mathbb{L}t}$$
$$Z_{\mathbb{P}^n}^{mot}(t) = \frac{1}{(1-t)(1-\mathbb{L}t)\cdots(1-\mathbb{L}^n t)}.$$
Completing $K_0(\mathrm{Var}_k)$ at $\mathbb{L}$, $[\mathrm{Sym}^n(\mathbb{P}^1)] = [\mathbb{P}^n]$ converge to
$$1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 + \cdots$$

[Similar statement for Grassmannians, flag varieties, affine algebraic groups, etc.]

3. Curves

Example 3 (Smooth proper algebraic curves of genus g w/ a rational point). There is a map $\pi_n : \mathrm{Sym}^n(C) \to \mathrm{Pic}^n(C)$ sending a divisor D to the line bundle $\mathcal{O}(D)$. Furthermore,
$$\pi^{-1}([\mathcal{L})]) \simeq \mathbb{P}\Gamma(C, \mathcal{L}).$$
[The points in the fiber $\pi^{-1}([\mathcal{L}])$ are exactly the divisors linearly equivalent to D; namely the roots of meromorphic functions with poles at D. These are precisely global sections of $\mathcal{O}(D)$, up to scaling.] By Riemann-Roch, and cohomology and base change, this is a (Zariski!) $\mathbb{P}^{n-g}$-bundle for $n \geq 2g-1$. [This requires a rational point for the representability of the Zariski Picard functor.] Thus
$$[\mathrm{Sym}^n(C)] = [\mathbb{P}^{n-g}][\mathrm{Jac}(C)]$$
for $n \geq 2g-1$.

Corollary 8 (Kapranov)**.** *Let C be a smooth proper curve with a rational point. Then*
$$Z_C^{mot}(t) = \frac{p(t)}{(1-t)(1-\mathbb{L}t)}$$
where $p(t)$ is a polynomial of degree $2g$. $Z_C^{mot}(t)$ satisfies a functional equation by Serre duality.
$$\lim_{n\to\infty}[\mathrm{Sym}^n(C)] = [\mathrm{Jac}(C)](1 + \mathbb{L} + \mathbb{L}^2 + \cdots)$$

in $K_0(\mathrm{Var}_k)$ *completed at* $(\mathbb{L})$.

For example, if E is an elliptic curve,

$$Z_E^{mot}(t) = \frac{1 + ([E] - [\mathbb{P}^1])t + \mathbb{L}t^2}{(1-t)(1-\mathbb{L}t)}.$$

[Note that this implies part of the Weil conjectures for curves. Kapranov leaves the hypothesis that C has a rational point implicit—however, we have]

Theorem 9 (L–). *Let* C *be a smooth proper genus* 0 *curve with no rational point—then* $Z_C(t)$ *is rational.*

[To my knowledge, the higher genus case is open—studying the Abel-Jacobi map for smooth proper curves with no rational points is pretty interesting, and related to the Brauer group of the Jacobian. Now that we've talked a bit about rationality for curves, let's talk about stabilization:]

Corollary 10. *In* $\widehat{K_0(\mathrm{Var}_k)}$ *[recall, this is the completion of the Grothendieck ring at* $(\mathbb{L})$*],*

$$\lim_{n\to\infty} [\mathrm{Sym}^n(C)] = [\mathrm{Jac}(C)](1 + \mathbb{L} + \mathbb{L}^2 + \cdots)$$

for C *a smooth proper curve with a rational point.*

[I'd like to draw analogies between this result and results in number theory and topology—I'll expand on these connections for the rest of the talk.]

Topology: Recall from the Dold-Thom theorem that

$$\mathrm{Sym}^\infty(C) \simeq_w K(H_1(C,\mathbb{Z}),1) \times K(H_2(C,\mathbb{Z}),2) \simeq K(\mathbb{Z}^{2g},1) \times K(\mathbb{Z},2).$$

Also, we have

$$\mathrm{Jac}(C) = H^0(C,\omega_C)^\vee / H_1(C,\mathbb{Z}) \simeq_w K(H_1(C,\mathbb{Z}),1)$$
$$\mathbb{CP}^\infty = \mathrm{pt} \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \cdots \simeq_w K(\mathbb{Z},2)$$

Thus the equality of Corollary 10 is an algebro-geometric analogue of the Dold-Thom theorem, for curves.

Number Theory: Let $C/\mathbb{F}_q$ be a smooth proper curve. Then if $\zeta_C(t)$ is the zeta function of C, we have

$$\mathrm{res}_{t=1}\, \zeta_C(t) = \frac{\#|\,\mathrm{Jac}(C)(\mathbb{F}_q)|}{1-q}.$$

[This is analogous to the analytic class number formula for zeta functions of number fields (e.g. the Beilinson-Lichtenbaum Conjectures).]

Similarly, we have

$$\mathrm{res}_{t=1}\, Z_C^{mot}(t) = \left[(1-t)Z_C^{mot}(t)\right]_{t=1} = \lim_{n\to\infty}[\mathrm{Sym}^n(C)] = [\mathrm{Jac}(C)](1+\mathbb{L}+\mathbb{L}^2+\cdots)\text{``}=\text{''}\frac{[\mathrm{Jac}(C)]}{1-\mathbb{L}}.$$

[So we can think of the Dold-Thom theorem of being a "topological" analogue to the analytic class number formula.]

4. Surfaces

[At first glance, many of these relationships seem to break down for algebraic surfaces, due to the "disappointing theorem" of Larsen and Lunts I discussed earlier—namely that the zeta functions of algebraic surfaces are in general not rational. In spite of this fact, I'd like to persuade you that in some ways, these power series behave as if they are rational or meromorphic. This is the "analytic number theory" aspect of the title of this talk.]

Suppose we are interested in understanding

$$\lim_{n\to\infty}[\mathrm{Sym}^n(X)]$$

for X a smooth projective surface. As before, this is formally equal to

$$[(1-t)Z_X(t)]_{t=1}$$

in any completion of $K_0(\mathrm{Var}_k)$. [This limit exists for rational and birationally ruled surfaces.] If $Z_X(t)$ were the power series of a meromorphic function, we would expect this limit to exist if its power series expansion were valid at $t = 1$—namely, if $Z_X(t)$ had no poles in the unit ball other than a pole of order one at $t = 1$. To understand this possibility, we need a small digression:

4.1. **The Newton polygon lies above the Hodge polygon.**

Definition 11 (Newton Polygon)**.** Let K be a non-archimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p(t)$). Let $p(t) = \sum_n a_n x^n \in K[t]$. Then the Newton polygon of p is the lower convex hull of the points $(i, v_K(a_i))$.

Theorem 12. *The slopes of the line segments appearing in the Newton polygon of p are (additiveley) inverse to the valuations of the roots of p (in its splitting field.)*

The relevance of this theorem is the following:

Theorem 13 (Grothendieck, Dwork, Deligne, Katz, Ogus, Mazur, ...)**.** *Let $X/\mathbb{Z}_p$ be a smooth projective variety of dimension n. Then*

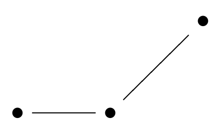
$$\zeta_{X_{\mathbb{F}_p}}(t) = \prod_{i=0}^{2n} p_i(t)^{(-1)^{i+1}}$$

where $p_i(t) \in \mathbb{Z}[t]$ is a polynomial of degree $b_i(X_{\mathbb{C}})$. The roots of the p_i have archimedean absolute value $q^{-i/2}$.

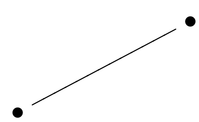
Furthermore, the Newton polygon of p_i with respect to the p-adic valuation lies above the degree i Hodge polygon of $X_{\mathbb{C}}$.

[This is far from the most general version of this theorem—also, I won't define the Hodge polygon, though I'll give some examples. The take-away, though, should be that the p-adic valuations of the roots and poles of $\zeta_{X_{\mathbb{F}_p}}$ are controlled by the Hodge theory of X.]

Example 4. Let $E/\mathbb{Z}_p$ be an elliptic curve. Its degree 1 Hodge polygon is

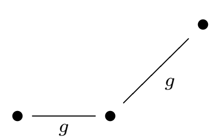


This is equal to its (degree 1) Newton polygon if $E_{\mathbb{F}_p}$ is ordinary; if it is super-singular, its Newton polygon is



[So in particular, these polygons are generically equal—indeed, the Newton polygon is generically equal to the Hodge polygon in families.]

Since we're interested in the "valuations" of "poles" of $Z_X^{mot}(t)$, I'd like to persuade you that this kind of analytic information is remembered by $Z_X^{mot}(t)$. I'll first state a theorem for curves of genus g; here the degree 1 Hodge polygon is:



Definition 14 ($\mathbb{L}$-adic Newton polygon)**.** Let $p(t) = \sum_n a_n t^n \in K_0(\mathrm{Var}_k)[t]$ be a polynomial. Then the $\mathbb{L}$-adic Newton polygon of p is the lower convex hull of $(i, v_{\mathbb{L}}(a_i))$, where $v_{\mathbb{L}}(x) \in \mathbb{Z}_{\geq 0} \cup \infty$ is the greatest integer n such that $x \in (\mathbb{L}^n)$.

Theorem 15 (L–)**.** *Let C be a smooth proper curve/k, with a rational point, and let $p(t)$ be the numerator of $Z_C^{mot}(t)$. Then the $\mathbb{L}$-adic Newton polygon of $p(t)$ lies above the degree 1 Hodge polygon of C; if $k = \bar{k}$ and $\mathrm{char}(k) = 0$, the two polygons are equal.*

[Note that this implies Katz-Mazur-Ogus if $k = \mathbb{F}_q$ and X is a curve.]

Proof. Riemann Roch and Serre duality for lies above; the characteristic zero statement is pretty interesting and relies on e.g. the fact that the theta divisor of a curve is not stably birational to its Jacobian. □

Now let's unwind this reasoning in the case of higher-dimensional varieties. We wish to know when $Z_X(t)$ has a pole of $\mathbb{L}$-adic valuation 0. Katz-Mazur-Ogus predicts this will happen if

$$h^0(X, \Omega_X^{2k}) \neq 0$$

for some k; for surfaces, this is precisely the non-vanishing of

$$h^0(X, \omega_X).$$

[So the failure of $\mathrm{Sym}^n(X)$ to converge in $\widehat{K_0(\mathrm{Var}_k)}$ should be taken as evidence that $Z_X^{mot}(t)$ "behaves like" a meromorphic function.] And indeed, we have

Theorem 16 (L–)**.** *Suppose* $\mathrm{char}(k) = 0$ *and let* X *be a smooth projective surface with* $h^0(X, \omega_X) > 0$. *Then*

$$\lim_{n\to\infty} [\mathrm{Sym}^n(X)]$$

does not exist in $\widehat{K_0(\mathrm{Var}_k)}$. *Furthermore, if either*

(1) $\mathbb{L}$ *is not a zero divisor in* $K_0(\mathrm{Var}_k)$, *or*
(2) $[X] = [Y]$ *implies* X *and* Y *are equidecomposable*

then MSSP (convergence in the completion of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ *at the dimension filtration) also fails for such* X.

Proof. Uses:

(1) Explicit desingularization of $\mathrm{Sym}^n(X)$ (the Hilbert scheme of n points on X) [constructing such desingularizations for $\dim(X) > 2$ is an important open problem].
(2) The birational geometry of $\mathrm{Sym}^n(X)$; in particular, rationally connected subvarieties must lie tangent to isotropic subspaces for 2-forms on $\mathrm{Sym}^n(X)$. [This is related to Bloch's conjecture about 0-cycles on surfaces.]

□

Remark 17. Melanie Matchett Wood has given an alternative proof of this result, using work of Donu Arupura.

In particular, our "analytic" heuristics for the class number formula for $Z_X^{mot}(t)$ have given the right prediction.

[I'd like to conclude with some open questions on related issues.]

(1) Is $Z_X^{mot}(t)$ meromorphic or rational in some sense? [In particular, it's rational in various specializations—is there a universal such specialization that can be explicitly described? To make this precise, I'd guess that $Z_X^{mot}(t)$ is rational over the Grothendieck ring of Chow motives.]
(2) Construct explicit desingularizations of $\mathrm{Sym}^n(X)$.
(3) Suggests conjectures on birational geometry of $\mathrm{Hilb}^n(X)$ for X a smooth projective surface.