Sunday, November 8, 2020 2:40 PM

André-Oort for products of modular curves

following "O-minimality and the Adré-Oort conjecture for $\mathbb{C}^n$" (2010, Pila)

Plan of this talk:

- state the main theorem
- explain the strategy of the proof
- give details

Theorem (Pila): Let $X = Y_1 \times \dots \times Y_n \times E_1 \times \dots \times E_m \times G^l$ where $n, m, l \geq 0$, $G = \mathbb{G}_m(\mathbb{C})$, $Y_i = \Gamma_i \backslash \mathbb{H}$ are modular curves with congruence subgroups Γ_i, and E_j are elliptic curves defined over $\overline{\mathbb{Q}}$. Suppose that V is a subvariety of X. Then V contains only a finite number of maximal special subvarieties.

Prior related work:

- Edixhoven (2005) had proved André-Oort for arbitrary (finite) products of modular curves assuming the Generalized Riemann Hypothesis for imaginary quadratic fields
- André (1998) had proved André-Oort for products of two modular curves unconditionally

Later related work:

- Tsimerman (2015) proved André-Oort for A_g (using a similar strategy)

*I will define special subvarieties after I explain the proof strategy

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Strategy: (vague)

initially proposed by Zanier (and fully worked out by Pila and Zannier to give a new proof of the Manin-Mumford)

- Step 1: define a transcedental uniformization $U=\mathbb{H}^n\times \mathbb{C}^m\times \mathbb{C}^l$ and map $\pi : U \longrightarrow X$, where $U = \mathbb{H}$ has certain properties including including the following: $\exists$ a discrete subgroup Γ of the group of isometries of U such that π is Γ-invariant. If $\mathbb{F}_X$ be a fundamental domain for the action of Γ on U, we let

$$\mathcal{Y} = \pi^{-1}(V), \quad Z = \mathcal{Y} \cap \mathbb{F}_X$$

terminology: pre-images of special pts of X will be called pre-special points.

- step 2: upper bound for the number of pre-special points of Z up to a given height that do not lie on some connected semi-algebraic set
  (key: Pila-Wilkie Theorem)

- step 3: lower number for the number of conjugates of a special point (and hence for the number of pre-special points in Z) in terms of the "complexity" of the special point.
  (key: siegel's thm + a result of Masser)

- step 4: Incompatibility of upper and lower bounds if the "complexity" is too large, unless Z contains semi-algebraic subsets ive dimension of positi

- step 5: identify the possible semi-algebraic subsets of $\underset{\sim}{\mathcal{Y}} = \pi^{-1}(V)$ (uses "functional transcedence")
- step 6: a final inductive argument

Special subvarieties of X (ADD)

Let $X = Y_1 \times \dots \times Y_n \times A \times \mathbb{G}^l$, where $Y_i = \Gamma_i \backslash \mathbb{H}$, and A is an abelian variety of dimension m defined over $\overline{\mathbb{Q}}$

- A special point of X is a point $(c,a,g) \in X$ such that c is a special point of $Y_1 \times \dots \times Y_n$, a is a special point of A, and g is a special point of $\mathbb{G}^l$.
- A special subvariety of X is a subvariety of the form $Y \times (a+B) \times gH$ where Y is a special subvariety of $Y_1 \times \dots \times Y_n$, is a special subvariety of A, and $a+B$ is a special subvariety of $\mathbb{G}^l$. gH is a special

recall: • A special point of $\mathbb{G}^l$ is a point $g = (g_1, \dots, g_l) \in \mathbb{G}^l$ of finite order (i.e. each g_i is a root of unity). By convention $\mathbb{G}^0$ is special

- A special subvariety of $\mathbb{G}^l$ is a subvariety of the form gH, where H is an algebraic subgroup and g is a torsion point in $\mathbb{G}^l$ (i.e $g = (g_1, \dots g_l)$ with g_i a root of unity)

recall: · A special point of A is a torsion point. By convention if A has dimension zero (which means that A consists of a single pt) then A is special.

- A special subvariety of A is a subvariety of the form $a+B$, where B is an abelian subvariety of A and a is a torsion point.

Def:

- A point on a modular curve $Y = \Gamma \backslash \mathbb{H}$ is called a special point of Y (or CM point of Y) if the corresponding elliptic curve has CM. A point on a product of modular curves is called a special point (or CM point) all its coordinates are special points

- Let $n \geq 0$. Let $S_0 \cup \dots\dots \cup S_k$ be a disjoint partition of $\{1, \dots, n\}$ with $k \geq 0$ and S_0 only permitted to be empty. Let $h_i \in \mathbb{H}$ be a point corresponding to a CM elliptic curve for each $i \in S_0$. Let s_i be the smallest element of S_i for each $i \geq 1$ and for each $j \in S_i$, $j \neq s_i$, choose $g_{ij} \in GL_2(\mathbb{Q})^+$. A Pre-special subvariety of $\mathbb{H}^n$ is a subvariety $\{(\tau_1, \dots, \tau_n) \in \mathbb{H}^n / \tau_i = h_i, i \in S_0, \tau_j = g_{ij}(\tau_{s_i}), i = 1, \dots, k, j \in S_i, j \neq s_i\}$

$N=\{(c_1,\dots$

for some choice of data as above.

Let now $Y_1\times\dots\times Y_n$, with $Y_i=\Gamma_i\backslash\mathbb{H}$ and π_i be the quotient map for each i. A special subvariety of $Y_1\times\dots\times Y_n$ is the image under $\pi_1\times\dots\times\pi_n$ of a pre-special subvariety of $\mathbb{H}^n$

$$\left(\text{recall: } \pi_i:\mathbb{H}\to\Gamma_i\backslash\mathbb{H}=Y_i \text{ so } \atop \pi_1\times\dots\times\pi_n:\mathbb{H}^n\to Y_1\times\dots\times Y_n\right)$$

Defining the map $\pi: U\to X$:

- If $Y_i=\Gamma_i\backslash\mathbb{H}$ then $U_i=\mathbb{H}$ and $\pi_i:\mathbb{H}\to Y_i$ is the quotient map.
- If A is an abelian variety of dim m. let $\Lambda\subseteq\mathbb{C}^m$ be a lattice such that $\mathbb{C}^m/\Lambda$ is complex analytically isomorphic to A. Then $U_A=\mathbb{C}^m$ and $\pi_A: U_A\to A$ is the composition of $\mathbb{C}^m\to\mathbb{C}^m/\Lambda$ with $\mathbb{C}^m/\Lambda\to A$. Let $\{\lambda_1,\dots\lambda_{2m}\}$ be a $\mathbb{Z}$-basis for Λ. $G_A=\mathbb{R}^{2m}$ where G_A acts on $\mathbb{C}^m$ by: if $g=(r_1,\dots r_{2m})\in G_A$ then $g(z)=z+\sum^{2m} r_i\lambda_i$. Then $\Gamma_A=G_A(\mathbb{Z})=\mathbb{Z}^{2m}$

and $\mathbb{F}_A = \{ \sum_{i=1}^{2m} t_i \lambda_i \ / \ 0 \le t_i < 1 \}$

- For G, $U_G = \mathbb{C}$, $\pi_G : U_G \to G$ exponential, G_G acts by: if $g = r \in \mathbb{R}$, $g(z) = z + 2\pi i r$, $\Gamma_G = G(\mathbb{Z}) = \mathbb{Z}$ and $\mathbb{F}_G = \{ z \in \mathbb{C} / 0 \le \operatorname{Im}(z) < 2\pi \}$

- For $X = Y_1 \times \dots \times Y_n \times A \times G^{\ell}$, $Y_i = \Gamma_i \backslash \mathbb{H}$

  $U = U_1 \times \dots \times U_n \times U_A \times (U_G)^{\ell}$, $\pi_X : U \to X$

  with $\pi_X = \pi_1 \times \dots \times \pi_n \times \pi_A \times (\pi_G)^{\ell}$

  $\Gamma_X = \Gamma_1 \times \dots \times \Gamma_n \times \Gamma_A \times (\Gamma_G)^{\ell}$, $\mathbb{F}_X = \mathbb{F}_1 \times \dots \times \mathbb{F}_n \times \mathbb{F}_A \times (\mathbb{F}_G)^{\ell}$

We put real coordinates on U (for U_A we must use lattice Λ) so that pre-special points are algebraic of bounded degree.

Prop: Let $X = Y_1 \times \dots \times Y_n \times A \times G^{\ell}$, $Y_i = \Gamma_i \backslash \mathbb{H}$ and A an abelian variety of dim $m \geq 0$. Then $\pi_X |_{\mathbb{F}_X}$ is definable in $\mathbb{R}_{an,exp}$

Upper bound:

Recall: A semi-algebraic set in $\mathbb{R}^v$ is the set of solutions to a finite collection of equations and inequalities ($<$, $\leq$) involving polynomials in $\mathbb{R}[x_1, \dots x_n]$.

Recall: Let $Z \subset \mathbb{R}^v$. $Z^{alg} = \bigcup_{\substack{W \text{ connected semi-algebraic} \\ \dim W > 0 \\ W \subset Z}} W$

Def: Let $P = (x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and assume that $x_0, \dots x_n \in K$ where K is a number field. The absolute multiplicative height of P is

$$H(P) = H_K(P)^{1/[K:\mathbb{Q}]} = \left(\prod_{v \in M_K} \max\{\|x_0\|_v, \dots \|x_n\|_v\} \right)^{1/[K:\mathbb{Q}]}$$

If $\alpha \in \overline{\mathbb{Q}}$, then we define

$$H(\alpha) = H(\alpha : 1).$$

For $Z \subset \mathbb{R}^v$, $k \geq 1$ integer and $T \geq 1$ real number let

$$Z(k,T) = \{ z = (z_1, \dots z_v) \in Z \,/\, \max_i [\mathbb{Q}(z_i) : \mathbb{Q}] \leq k, \max_i H(z_i) \leq T \}$$

where $H(\alpha)$ is the absolute height of an alg.

number.

set $N_k(Z,T) = \# Z(k,T)$

Thm (Pila-Wilkie, Pila)

Let $Z \subset \mathbb{R}^v$ be definable, $k \geq 1$ and $\varepsilon > 0$. Then $\exists\, c(Z,k,\varepsilon)$ such that $\forall T \geq 1$

$$N_k(Z \setminus Z^{alg}) \leq c(Z,k,\varepsilon)\, T^{\varepsilon}$$

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lower bound:

For a complex quadratic $\tau \in \mathbb{H}$ we have that τ is the root of a unique polynomial $a\tau^2 + b\tau + c$, $a,b,c \in \mathbb{Z}$, $(a,b,c) = 1$, $a > 0$. the discriminant $D(\tau)$ of τ is $b^2 - 4ac$

Def: Let $X = Y_1 \times \dots \times Y_n \times E_1 \times \dots \times E_m \times \mathbb{G}_m^l$, $Y_i = \Gamma_i \backslash \mathbb{H}$. let $u = (\tau_1, \dots \tau_n, z_1, \dots z_m, \mathfrak{z}_1, \dots \mathfrak{z}_l) \in U_X$ be a pre-special point. Let D_i be the discriminant of τ_i, $i = 1, \dots, n$, let T be the order of the image of $(z_1, \dots, z_m)$ in $\mathbb{C}^m / \Lambda_1 \oplus \Lambda_2 \oplus \dots \oplus \Lambda_m$ and let N be the order of the image of $(\mathfrak{z}_1, \dots, \mathfrak{z}_l)$ in $(\mathbb{C}/2\pi i \mathbb{Z})^l$. The complexity of u is defined by

$$\Delta(u) = \max(|D_1|, \dots |D_n|, T, N)$$

note: if $B > 0$, $\exists$ only finitely many special points of X corresponding to pre-special points u with $\Delta(u) \le B$

Prop: Let $X = Y_1 \times \dots \times Y_n \times E_1 \times \dots \times E_m \times G^l$. $\exists\ C_{\text{height}}(X) > 0$ such that if $u = (\tau_1, \dots \tau_n, z_1, \dots z_m, \mathfrak{z}_1, \dots \mathfrak{z}_l) \in \mathbb{F}_X$ is a prespecial point. Then

$$H(u) \le C_{\text{height}}(X)\Delta(u)$$

Prop: Let $X = Y_1 \times \dots Y_n \times E_1 \times \dots E_m \times G^l$. $\exists\ C_{\text{degree}}(X) > 0$ such that if $u \in U$ is a pre-special point, then

$$[\mathbb{Q}(\pi(u)) : \mathbb{Q}] \ge C_{\text{degree}}(X)\,\Delta(u)^{1/7}$$

sketch of pf:

Write $u = (\tau_1, \dots \tau_n, z_1, \dots z_m, \mathfrak{z}_1, \dots \mathfrak{z}_l)$

$$[\mathbb{Q}(j(\tau_i)) : \mathbb{Q}] = h(D(\tau_i)) \ge C_{\text{siegel}}(v)\,|D(\tau_i)|^{1/2 - v}$$

(Siegel's thm)

Since Y_i is a finite cover of $\mathbb{C}$, we get a similar lower bound up to some constant depending on Y_i.

By results of Masser if $P_i \in E_i$ is the image of z_i, then $[\mathbb{Q}(P_i) : \mathbb{Q}] \ge C(E_i)\,T^{1/7}$

and finally

$$\frac{\varphi(n)}{n^{1-v}} \to \infty \quad (\text{as } n \to \infty)$$

Putting everything together we have proved the proposition $\square$

Putting everything together:

Def: Let $V \subset X$. The special set of V, denoted V^{sp}, is
$$V^{sp} = \bigcup_{\substack{W \text{ special} \\ \text{subvariety} \\ \text{of } V, \\ \dim(W) > 0}} W .$$

note: · If V^{sp} consists of a finite union of special subvarieties then it is an algebraic variety

· If X, V are defined over a number field K then V^{sp} is defined over K (key: the conjugates over K of a special subvarieties are special subvarieties)

Thm: Suppose $X = Y_1 \times \dots \times Y_n \times E_1 \times \dots \times E_m \times G^l$ and that V is a subvariety of X defined over a number field K that contains a field of definition for X. Suppose also that V^{sp} is a variety. Then $V \setminus V^{sp}$ contains only finitely many special points.

Pf:

Let $\tilde{Z} = \pi^{-1}(V)$ and $\tilde{Z}^{ps} = \pi^{-1}(V^{sp})$.

Put $Z^{ps} = \tilde{Z}^{ps} \cap F_X$.

Let $N_g^{\text{prespecial}}(W, T)$ be the number of pre-special points in a set W up to height T.

$\forall \varepsilon > 0, T \geq 1$ we have

$$N_g^{\text{prespecial}}(Z \setminus Z^{ps}, T) \overset{(*)}{=} N_g^{\text{prespecial}}(Z \setminus Z^{alg}, T) \leq$$

$$\leq N_g(Z \setminus Z^{alg}, T) \leq c(Z, g, \varepsilon) T^{\varepsilon}$$

(I will not explain why $(*)$ but it follows from work of Pila in the paper)

Suppose now that $Z \setminus Z^{ps}$ contains a pre-special point of complexity $\Delta = \Delta(u)$. Then $x = \pi(u) \in V \setminus V^{sp}$ is special and has at least

$$\frac{C \text{degree}(X) \Delta^{1/7}}{[k : \mathbb{Q}]}$$

conjugates x' which also lie in $V \setminus V^{sp}$. These conjugates have distinct pre-images $u' \in Z \setminus Z^{ps}$, having complexity $\Delta(u') = \Delta(u) = \Delta$, and hence $H(u') \leq C \text{height}(X) \Delta$.

Put $T = c_{height}(X)\Delta$.

Then

$$\frac{c_{degree}(X)}{[K:\mathbb{Q}]\, c_{height}(X)^{1/7}}\, T^{1/7} \le N_2(Z \setminus Z^{sp}, T) \le c(Z, 2, \varepsilon) T^{\varepsilon}$$

choose $\varepsilon = 1/8$. From the above inequalities Δ must be bounded $\Rightarrow$ the special points of $V \setminus V^{sp}$ come from a finite set $\square$

Thm: Let $X = Y_1 \times \dots \times Y_n \times E_1 \times \dots \times E_m \times G_1^{l}$, $Y_i = \mathbb{H}/\Gamma_i$ and E_i are defined over $\overline{\mathbb{Q}}$. Suppose that V is a subvariety of X. Then V contains only a finite number of maximal special subvarieties.

Pf:

$\exists$ a subvariety $\overline{V} \subset V$, defined $/\overline{\mathbb{Q}}$, that contains all the algebraic points of V. So we may assume that V is defined over $\overline{\mathbb{Q}}$.

We use induction on $\dim X$.

If $\dim X = 1$ then $V = X$ or $V =$ a finite set of points. ✓

If $\dim X = 2$ and $V \neq X$ then $\dim V \le 1$ and can contain only finitely many components of dimension 1, so V^{sp} is a subvariety and the thm follows by the previous thm.

Assume that $\dim X \geq 3$, using the inductive hypothesis (details omitted) we can show that V^{sp} is a subvariety so by the previous thm the theorem follows. $\square$