Tiling Problems

Daniel Litt

Stanford University

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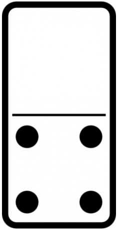
Introduction

Tiles:



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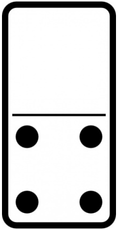

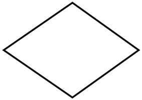

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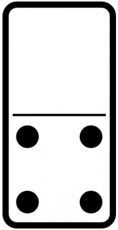

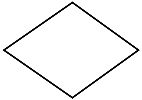

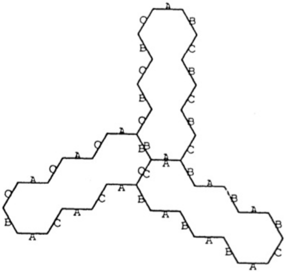

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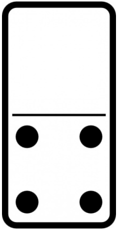

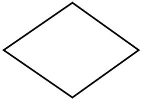

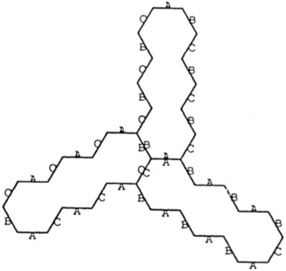

Definition (Tile)

*A **tile** is a (closed) plane polygon.*

Introduction (cont.)

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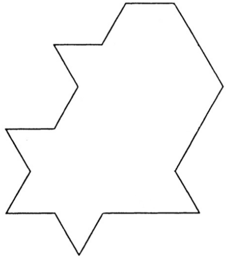

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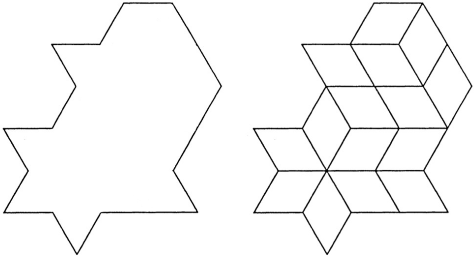

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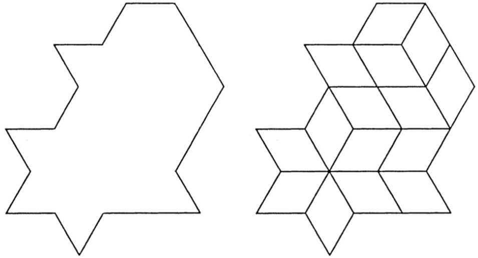

Definition (Region)

A **region** *is a (closed) plane polygon.*

Introduction (cont.)

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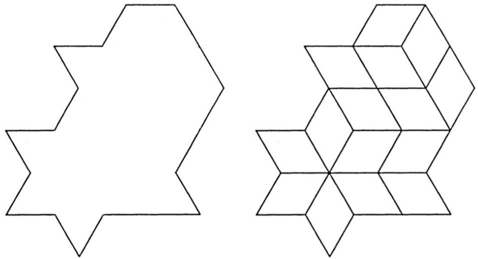

Definition (Tiling)

A tiling of a region R is a decomposition of R into tiles, $R = \bigcup_i T_i$, such that if x is a point in the interior of a tile T_i, then it is not contained in any T_j for $j \neq i$.

Introduction (cont.)

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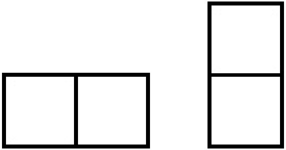
    - Given a set of tiles, can one tile the plane with them? This is undecidable.

Counting Problems (warmup)

Tiles: 2×1 rectangles (dominos):



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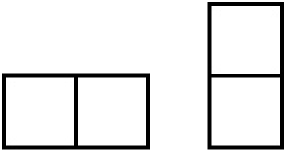
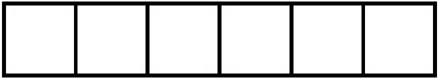


Region: $1 \times n$ grid.



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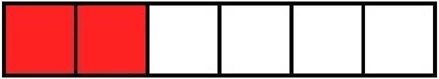

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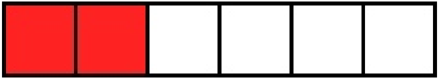

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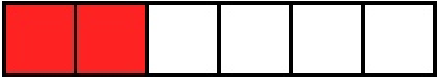

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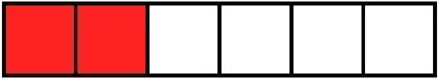

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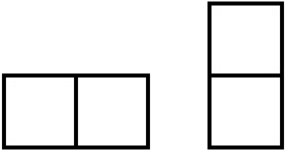
$$T_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

We'll return to this example.

Counting Problems (trickier example)

Tiles: 2×1 rectangles (dominos):



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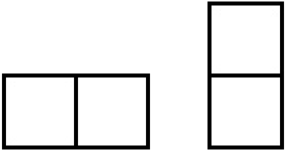
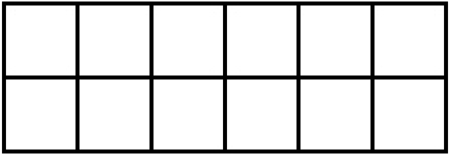


Region: $2 \times n$ grid.



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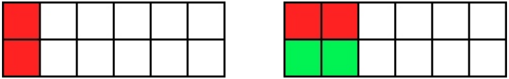

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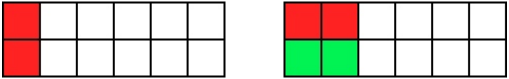

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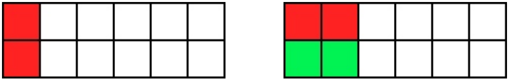

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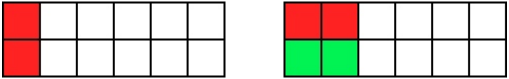

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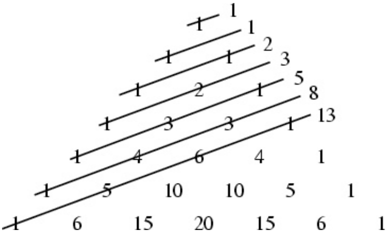
Fibonacci numbers!

Applications (Fibonacci identities)

- $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots = f_n$:



- $\sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j}\binom{n-j}{i} = f_{2n+1}$
- For $m \geq 1, n \geq 0$, if $m|n$ then $f_{m-1}|f_{n-1}$.
- $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$

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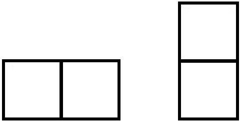

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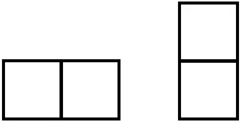

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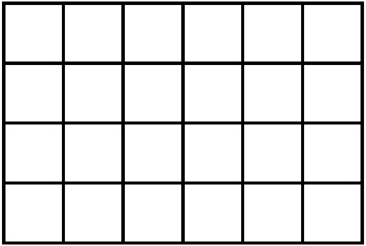

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A: Pfaffians!

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Definition (Pfaffian)

Let Π *be the set of partitions of* $\{1, 2, ..., 2n\}$ *into pairs*

$$\alpha = \{(i_1, j_1), (i_2, j_2), ..., (i_n, j_n)\}$$

with $i_k < j_k$ *and* $i_1 < i_2 < i_3 < \cdots < i_n$. *The* **Pfaffian** *of* A *is defined to be*

$$\mathrm{pf}(A) = \sum_{\alpha \in \Pi} \mathrm{sign}(\alpha) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

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Theorem

$$\mathrm{pf}(A) = \pm\sqrt{\det(A)}$$

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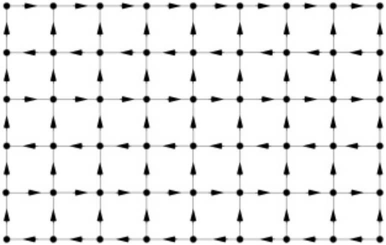
Remark

If A is the adjacency matrix of an oriented graph, $\mathrm{pf}(A)$ *counts oriented perfect matchings.*

Counting Problems (dominos, $m \times n$ case)



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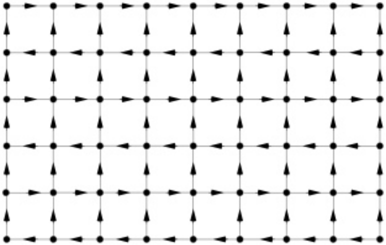

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So $T_{m,n} = \text{sqrt}(\det(A))$. Compute by diagonalizing A.

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Remark

In general, any planar graph has a "Pfaffian Orientation," which makes the above argument work.

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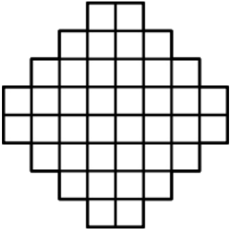
But this isn't the end of the story. Consider the aztec diamond:



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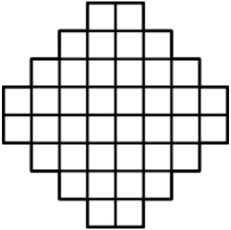

Number of tilings is $2^{(n+1)n/2}$. But add one more row in the middle, and the number of tilings only grows exponentially.

Feasibility Problems

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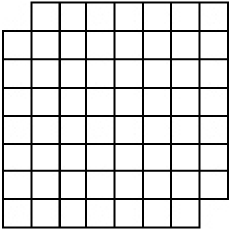
- Given a set of tiles, can they tile the plane? This is undecidable.

A Classical Example

Can this region be tiled by dominos?



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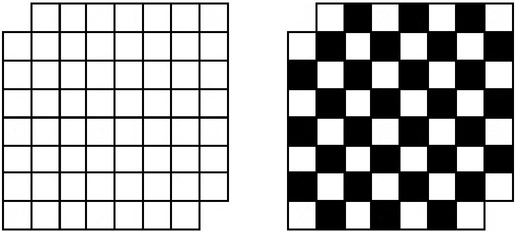

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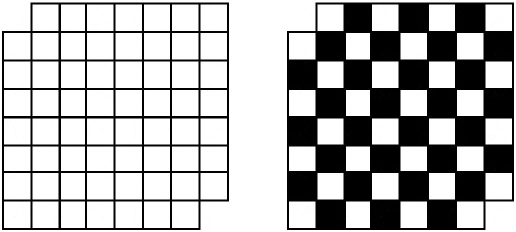

Each domino covers exactly one black square and one white square; but there are more white squares than black squares.

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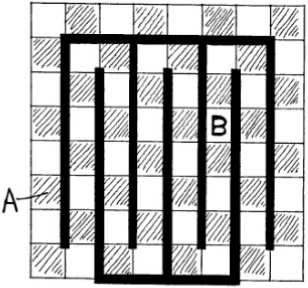

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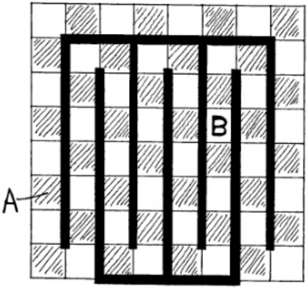
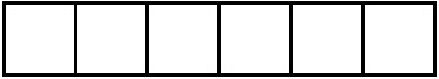


(Gomory)
Of course, if we remove more than 2 squares, a lot can go wrong.

Rectangle Tilings (Toy Example)

Let's return to tiling a $1 \times n$ rectangle R_n by dominos.



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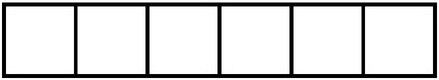

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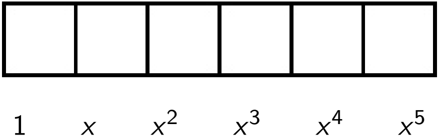

$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$
$d(x) = 1 + x$
Note that $d(-1) = 0$.

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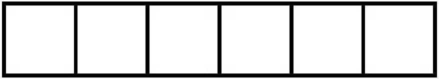

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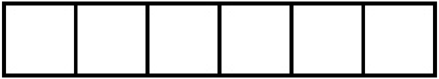

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But $p_{2n+1}(x)$ is not a multiple of $d(x)$:

$$p_{2n+1}(-1) = 1$$

Rectangle Tilings

Label the upper-right quadrant of the plane as follows:

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\hline
\multicolumn{1}{|c|}{y^2} & \multicolumn{1}{c|}{xy^2} & \multicolumn{1}{c|}{x^2y^2} & \multicolumn{1}{c|}{x^3y^2} & \cdots \\
\hline
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If R is a region consisting of unit squares (α, β) with non-negative integer coordinates, let

$$p_R(x, y) = \sum_{(\alpha,\beta)\in R} x^\alpha y^\beta.$$

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y	xy	x^2y	x^3y	$\cdots$
1	x	x^2	x^3	$\cdots$

If R is a region consisting of unit squares (α, β) with non-negative integer coordinates, let

$$p_R(x, y) = \sum_{(\alpha,\beta)\in R} x^\alpha y^\beta.$$

If T_i are tiles made from unit squares, translate them so one square is at the origin, and let

$$p_{T_i}(x, y) = \sum_{(\alpha,\beta)\in T_i} x^\alpha y^\beta.$$

Rectangle Tilings

If R may be tiled by the T_i then there exist polynomials $a_i(x, y)$ with integer coefficients such that

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Definition

If there exist polynomials $a_i(x, y)$ with coefficients in a ring k, such that

$$p_R(x, y) = \sum_i a_i(x, y) p_{T_i}(x, y),$$

we say that the T_i can tile R over k.

Rectangle Tilings

Theorem

Let T_i be a (possibly infinite) set of tiles. Then there exists a finite subset T_{i_j} such that a region R may be tiled by the T_i over the integers if and only if it may be tiled by the T_{i_j}.

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Proof.

Hilbert Basis Theorem. □

Rectangle Tilings

Let $k = \mathbb{C}$, the complex numbers. Let $V \subset \mathbb{C}^2$ be the set

$$V = \{(x, y) \mid T_i(x, y) = 0 \text{ for all } i\}.$$

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Theorem

Let I_T be the set of all polynomials that can be written as in $(*)$. If I_T is **radical**, *and $p_R(x, y) = 0$ for all $(x, y) \in V$, then the T_i may tile R over $\mathbb{C}$.*

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Theorem (Barnes)

Let T be a finite set of rectangular tiles, and R a rectangular region. There exists a constant K such that if the lengths of the sides of R are greater than K, then R is tileable by T if and only if it is tileable over $\mathbb{C}$.

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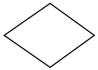
$\vdots \quad \vdots \quad \vdots \quad \vdots$

$+1$	-1	$+1$	$+1$	$\cdots$
-1	$+1$	-1	$+1$	$\cdots$
$+1$	-1	$+1$	$+1$	$\cdots$

Lozenge Tilings

This is a lozenge:



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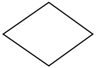

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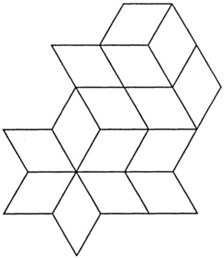
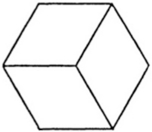


Squint a little:



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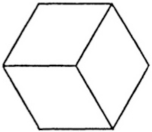
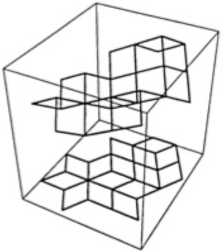


Pick a direction for each edge. Does the outline of a region lift to a loop?



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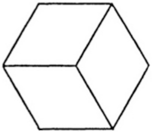

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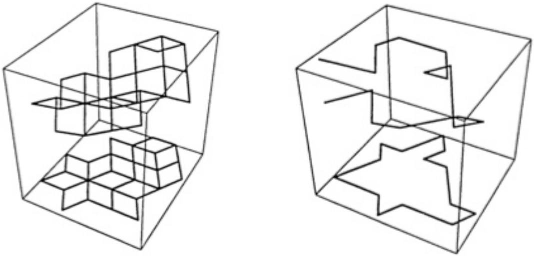
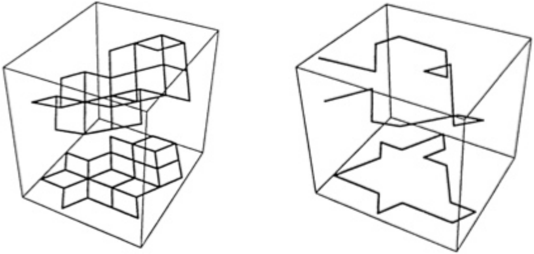
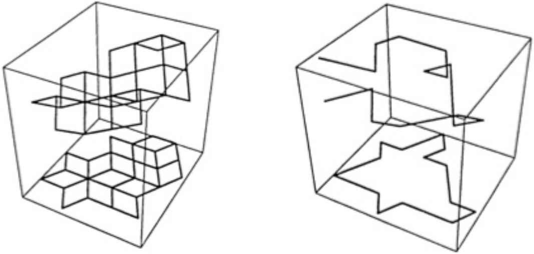




This condition is necessary but not sufficient; there is a sufficient geometric condition (Conway, Thurston).



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This method generalizes, but is difficult to analyze except in special cases.

Open Problems

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- And more...