Structure of T_m (Serre l-adic reps 2.3)

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Setup

- K = number field
- v = place of K $\quad\left(\Sigma_K = \Sigma_K^{fin} \cup \Sigma_K^{\infty} \right.$ = set of places$\left.\right)$
- I = ideles of K $\quad$ [restricted product $\prod_{v \in \Sigma_K^{fin}} (K_v^\times, \mathcal{O}_v^\times) \times \prod_{v \in \Sigma_K^\infty} K_v^\times$]
- S = finite subset of Σ_K^{fin}.
- $\mathfrak{m} = \sum_{v \in S} m_v \cdot [v]$ $\quad m_v \in \mathbb{Z}_{>0}$ $\quad$ "modulus with support S"
- $U_{\mathfrak{m}} \subset I$ is $\left\{ (x_v)_v \in I : \begin{array}{l} v(x_v - 1) \geq m_v \text{ for all } v \in S \\ x_v > 0 \text{ for } \underline{\text{real}}\ v \in \Sigma_v^\infty \end{array} \right\}$
- $I_{\mathfrak{m}} := I / U_{\mathfrak{m}}$

$E_{\mathfrak{m}} := \mathcal{O}_K^\times \cap U_{\mathfrak{m}}$

- $$E_{\mathfrak{m}} = \left\{ \begin{array}{l} \text{units} \\ x \in \mathcal{O}_K^\times \end{array} : \begin{array}{ll} v(x-1) \geq m_v & \text{for } v \in S, \\ v(x) > 0 & \text{all all real places } v \in \Sigma_K^\infty \\ \multicolumn{2}{l}{\text{no condition at complex } v \text{ or } v \in \Sigma_K^{\text{fin}} \setminus S.} \end{array} \right\}$$

- $E_{\mathfrak{m}}$ = units that are "close" to 1 at places in S.
- $E_{\mathfrak{m}}$ is <u>finite index</u> in $\mathcal{O}_K^\times$ ⇒ "arithmetic subgroup".

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ray class group.

$$1 \longrightarrow K^\times / E_{\mathfrak{m}} \longrightarrow I_{\mathfrak{m}} \longrightarrow C_{\mathfrak{m}} \longrightarrow 1$$

- Set $T = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$. $T(A) = \mathbb{G}_m(K \otimes A) = (K \otimes A)^\times$.
- Interpret $K^\times$ as $T(\mathbb{Q})$
- $E_{\mathfrak{m}} \subset T(\mathbb{Q})$, $\overline{E_{\mathfrak{m}}}$ = Zariski closure
- $T_{\mathfrak{m}} := T / \overline{E_{\mathfrak{m}}}$
- Get map $K^\times / E_{\mathfrak{m}} \longrightarrow T_{\mathfrak{m}}(\mathbb{Q})$.

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$$\begin{tikzcd} 1 \arrow[r] & K^\times / E_{\mathfrak{m}} \arrow[r] \arrow[d] & I_{\mathfrak{m}} \arrow[r] \arrow[d] & C_{\mathfrak{m}} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & T_{\mathfrak{m}}(\mathbb{Q}) \arrow[r] & S_{\mathfrak{m}}(\mathbb{Q}) \arrow[r] & C_{\mathfrak{m}} \arrow[r] & 1 \end{tikzcd}$$

Last Time: $\mathrm{Gal}(K^{ab}/K) \cong \underbrace{(I/K^\times)}_{\text{Idele class group}} / \text{connected component}$

$\rightsquigarrow \varepsilon_\ell : \mathrm{Gal}(K^{ab}/K) \longrightarrow S_m(\mathbb{Q}_\ell)$

Goal: Understand ℓ-adic rep'ns of $\mathrm{Gal}(K^{ab}/K)$ via ℓ-adic rep'ns of S_m.

So, given any $\phi : S_{m,\mathbb{Q}_\ell} \longrightarrow GL(V_\ell)$,

get an ℓ-adic representation $\phi_\ell = \phi \circ \varepsilon_\ell : \mathrm{Gal}(K^{ab}/K) \to GL(V_\ell)$

Properties
(a) ϕ_ℓ is semi-simple (since S_m is of multiplicative type)
(b) ϕ_ℓ is unramified outside $S \cup \{\mathfrak{p} \in \Sigma_K^{fin} : \mathfrak{p} \mid \ell\}$ — same for ε_ℓ.
and we know $\phi_\ell(\mathrm{Fr}_v)$!
(c) $\underbrace{\phi_\ell \text{ is rational}}_{\text{char poly}(\phi_\ell(\mathrm{Fr}_v)) \in \mathbb{Q}[T] \text{ for } v \notin S \cup \{\mathfrak{p} \mid \ell\}} \Leftrightarrow \phi$ can be defined over $\mathbb{Q}$ — used Čebotarev

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Thm 2.5

If $\phi : S_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$ comes from $\phi_0 : S_m \to GL(V_0)$ over $\mathbb{Q}$

(1) $\{\phi_\ell\}$ are a strictly compatible system of rational, semi-simple, abelian Galois rep'ns.
ⓒ ⓐ (b bounds exceptional set)

(3) $\exists \infty$'ly many ℓ such that ϕ_ℓ is diagonalizable over $\mathbb{Q}_\ell$.

(2) and we know $\phi_\ell(\mathrm{Fr}_v)$

Proof of (3)

(i) ϕ_0 is diagonalizable over some number field E.

(ii) If ℓ splits completely in E, then ϕ splits over $E_\lambda = \mathbb{Q}_\ell$.

(iii) Cebotarev $\Rightarrow$ infinitely many ℓ split completely.

Remark: All eigenvalues of Frobenius are contained in E.

$$\begin{tikzcd}
 & K^\times \arrow[rr] \arrow[dr] \arrow[dd] & & I \arrow[dr] \arrow[dddr, "\varepsilon_m"'] & & & \\
1 \arrow[rr] & & K^\times/E_m \arrow[rr] \arrow[dd] & & I_m \arrow[r] \arrow[dd] & C_m \arrow[r] \arrow[dd, equal] & 1 \\
 & T(\mathbb{Q}) \arrow[dr] & & & & & \\
1 \arrow[rr] & & T_m(\mathbb{Q}) \arrow[rr] & & S_m(\mathbb{Q}) \arrow[r] & C_m \arrow[r] & 1
\end{tikzcd}$$

$F = \phi \circ \varepsilon_m$

$\phi : S_m \to GL_{V_0}$ is a map of algebraic groups.

Converse (sort of)

A homomorphism $f : I \to GL(V_0)$ factors through $S_m(\mathbb{Q})$ if and only if

(a) $\ker(f) \supset U_m$, and

(b) $\exists\, \psi : T \to GL_{V_0}$ (morphism of algebraic groups) s.t. $\psi(x) = \phi(x)$ for all $x \in K^\times$.

The factorization is unique.

Proof (a) + (b) give compatible maps $I_m \to GL_{V_0}$ and $T_m \to GL_{V_0}$ + univ. prop. of S_m.

The point: Given a rational ℓ-adic representation

(Goal of future talks) $\rightsquigarrow$ Comes from some $S_m \longrightarrow GL_{V_0}$

$\rightsquigarrow$ Get a whole compatible system $\{G^{ab} \to GL(V_0 \otimes \mathbb{Q}_{\ell'})\}$

Part II : Example: ℓ-adic Tate module of a CM abelian variety.

A/K abelian variety of dimension d (elliptic curve)

E number field of degree $2d$ (imaginary quadratic)

If $\exists\, \iota : E \hookrightarrow \mathrm{End}_K(A) \otimes \mathbb{Q}$, say A has complex multiplication by E

$$V_\ell(A) = T_\ell(A) \otimes \mathbb{Q} = \left(\varprojlim_{\ell} A[\ell^n]\right) \otimes \mathbb{Q}.$$

V_ℓ is a free $\mathbb{Q}_\ell$-module of dimension $2d$

V_ℓ is a free $E \otimes \mathbb{Q}_\ell$-module of dimension 1.

$\mathrm{Gal}(\bar{K}/K) \circlearrowright V_\ell$ induces $\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to (E \otimes \mathbb{Q}_\ell)^\times$
$$= (\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m)(\mathbb{Q}_\ell).$$

Set $T_E = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$

<u>Theorem</u> 1. The (ρ_ℓ) are a strictly compatible system of ℓ-adic reps of $\mathrm{Gal}(K^{ab}/K)$ with values in T_E.

2. $\exists$ modulus $\mathfrak{m}$, $\phi : S_\mathfrak{m} \to T_E$ s.t. $\rho_\ell = \phi \circ \varepsilon_\ell$ ← Canonical system.

Can compute composition $T \to T_\mathfrak{m} \to S_\mathfrak{m} \to T_E$ using tangent space of A!

<u>Example</u> A = elliptic curve w/ CM over E = imaginary quadratic

tangent space $T_0(A) \cong K$ has E and K v-space structures.

action of E gives $E \hookrightarrow \mathrm{End}_K(T_0(A)) = K$

action of K gives
$$\underbrace{K \to \mathrm{End}_E(T_0(A)) \xrightarrow{\det_E} E}_{\mathrm{Nm}_{K/E} : K \to E \text{ relative to embedding.}}$$

T_E

induces

$$\begin{array}{ccc} \mathrm{Res}_{K/\mathbb{Q}}\, \mathbb{G}_m & \longrightarrow & \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m \\ \| & & \| \\ T & & T_E. \end{array}$$

Section III: Structure of $T_{\mathfrak{m}}$

Goal: Understand linear rep'ns $S_{\mathfrak{m}} \longrightarrow GL_{V_0}$.

Really just need to understand $T_{\mathfrak{m}}$.

Question: What is $T / \overline{(\text{arithmetic subgroup})}$?

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Example: $K = \mathbb{Q}(\sqrt{-a}, \sqrt{-b})$ $\quad$ $a, b > 0$, don't differ by a square

$$\operatorname{rk} \mathcal{O}_K^\times = \underset{\substack{\uparrow \\ \text{real} \\ \text{places}}}{0} + \underset{\substack{\uparrow \\ \text{cx} \\ \text{places.}}}{2} - 1 = 1$$

positive rank all comes from $\mathcal{O}_{\mathbb{Q}(\sqrt{ab})}^\times$.

isogenous $\quad$ $x^2 + ay^2 = 1$ $\quad$ rank 0 $\quad$ $x^2 + by^2 = 1$ $\quad$ rank 0

\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{-a}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{-b}) \arrow[d, dash] & \mathbb{Q}(\sqrt{ab}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}

Res $\mathbb{G}$

$$\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m \sim \left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m \times \left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m$$

$K = 1\cdot\mathbb{Q} \oplus \sqrt{-a}\cdot\mathbb{Q} \oplus \sqrt{-b}\cdot\mathbb{Q} \oplus \sqrt{ab}\cdot\mathbb{Q}.$

$$\times \underbrace{\left(\operatorname{Res}_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m}_{\text{rank } 1} \times \underbrace{\mathbb{G}_m}_{\text{rank } 0.}$$

$\boxed{\mathbb{Z}\text{-points}}$

$$\begin{tikzcd}
 & \operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m & \\
\operatorname{Res}_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}} \mathbb{G}_m \arrow[ur, leftrightarrow] & \operatorname{Res}_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}} \mathbb{G}_m \arrow[u, leftrightarrow] & \operatorname{Res}_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}} \mathbb{G}_m \arrow[ul, leftrightarrow] \\
 & \mathbb{G}_m \arrow[ul, leftrightarrow] \arrow[u, leftrightarrow] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

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quotient by closure of arithmetic subgroup kills piece of positive rank.

$$\operatorname{Res}_{K/\mathbb{Q}} \mathbb{G}_m \Big/ \overline{\left(\begin{matrix}\text{arithmetic}\\ \text{subgroup}\end{matrix}\right)} \sim \left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m \times \left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m \times \mathbb{G}_m.$$

Fancier argument (Sketch): Suppose $T/\mathbb{Q}$ irreducible torus, E = arithmetic subgroup.

Survive Quotient ☆ 1. If $T \cong \mathbb{G}_m$, then E is finite, $T/\overline{E} \sim T$.

Thm (Ono) 2. If $T \not\cong \mathbb{G}_m$ (over $\mathbb{Q}$), then $T(\mathbb{R})/E$ is compact.

Dies in the quotient ✗ (a) If T is not compact $\Rightarrow$ E is infinite $\Rightarrow$ $\overline{E} = T$, $T/\overline{E} = \{0\}$

Survive quotient ☆ (b) If T is compact $\Rightarrow$ E is finite $\Rightarrow$ $T/\overline{E} \sim T$.

3. $T(\mathbb{R})$ is compact $\Leftrightarrow$ (up to isogeny) $T \subset \left(\operatorname{Res}_{L/\mathbb{Q}} \mathbb{G}_m\right) \big/ \left(\operatorname{Res}_{L_0/\mathbb{Q}} \mathbb{G}_m\right)$

for some L_0 totally real and $[L:L_0] = 2$, L totally complex.

$L = \mathbb{Q}(\sqrt{-a}, \sqrt{-b})$ $L_0 = \mathbb{Q}(\sqrt{ab})$

$$\underbrace{\left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m}_{\text{anisotropic, compact}} \times \underbrace{\left(\operatorname{Res}_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m}_{\text{anisotropic, compact}} \times \underbrace{\cancel{\left(\operatorname{Res}_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}} \mathbb{G}_m\right)/\mathbb{G}_m}}_{\text{anisotropic, noncompact}} \times \underbrace{\mathbb{G}_m}_{\text{isotropic.}}$$

Studying action of Galois on the group of characters of $T_{\overline{\mathbb{Q}}}$ gives a recipe to break up any T as

$$T \sim \underset{\substack{\uparrow \\ \text{isotropic}}}{T_0} \times \underset{\substack{\uparrow \\ \text{anisotropic,} \\ \text{compact}}}{T^-} \times \underset{\substack{\uparrow \\ \text{anisotropic,} \\ \text{no compact factors.}}}{T^+}$$

Then $$T/\overline{E} \sim T_0 \times T^- .$$

<u>Upshot</u>: Let $K_0 \subset K$ be the maximal totally real subextension of K.

If $\exists\ K_1 \subset K$, $[K_1 : K_0] = 2$, K_1 tot. imaginary, then

$$T_{\mathfrak{m}} \sim \underset{1}{\mathbb{G}_m} \times \underbrace{\left(\mathrm{Res}_{K_1/\mathbb{Q}}\, \mathbb{G}_m\right) / \left(\mathrm{Res}_{K_0/\mathbb{Q}}\, \mathbb{G}_m\right)}_{[K_0 : \mathbb{Q}]} .$$

$+$

Otherwise,

$$T_{\mathfrak{m}} \sim \underset{1}{\mathbb{G}_m}.$$