AUTOMORPHIC FORMS NOTES, PART I

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The goal of these notes are to take the classical theory of modular/automorphic forms on the upper half plane and reinterpret them, first in terms $L^2(\Gamma \setminus SL(2,\mathbb{R}))$ and then in terms of $L^2(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}}))$.

1. The Classical Theory, Super-Quick Review

Let Γ be an arithmetic subgroup of $SL(2,\mathbb{R})$, $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ the upper half plane, $GL^+(2,\mathbb{R})$ the group of real 2×2 matrices with positive determinant. We fix the following conventions:

- $GL^+(2,\mathbb{R})$ acts on $\mathbb{H}$ via fractional linear transformations
$$g(z) = \frac{az+b}{cz+d}$$
for
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
- The *factor of automorphy* j is defined as
$$j(g,z) := (cz+d)(\det(g))^{-1/2}.$$
- For f a complex-valued function on $\mathbb{H}$,
$$f|_{[g]_k}(z) := f(gz)j(g,z)^{-k}.$$
- Let $\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps of } \Gamma\}$. Then the action of Γ on $\mathbb{H}$ extends to an action on $\mathbb{H}^*$, and $\Gamma \setminus \mathbb{H}^*$ may be viewed naturally as a compact Riemann surface.

Definition 1 (Automorphic Form of Weight k)**.** *A holomorphic function on $\mathbb{H}$ is a Γ-automorphic form of weight K if*

(1) $f|_{[\gamma]_k} = f$
(2) *f is holomorphic at every cusp of Γ.*

The space of such functions is called $M_k(\Gamma)$.

Note that if σ sends some ∞ to a fixed cusp s, then $f|_{[\sigma]_k}(z)$ is periodic with some period h , and so has a Fourier expansion $\hat{f}_s(e^{2\pi iz/h}) = f|_{[\sigma]_k}(z)$; the Taylor series for $\hat{f}$ is the Fourier expansion of f at s. Holomorphicity at cusps is the statement that $\hat{f}$ has a removable singularity at the origin.

Definition 2 (Cusp Form)**.** *A Γ-automorphic form is a* cusp form *if it vanishes at every cusp of Γ, in the sense just defined. The space of Γ-cusp forms of weight k is denoted $S_k(\Gamma)$.*

2. Reintepretation in terms of $L^2(\Gamma \setminus SL(2,\mathbb{R}))$

Let $G = SL(2,\mathbb{R}), \Gamma = \Gamma_0(N)$. We wish to identify $S_k(\Gamma)$ with a finite-dimensional subspace of $L^2(\Gamma \setminus G)$. Now G acts transitively on $\mathbb{H}$, so we simply let $\phi : S_k \to L^2(\Gamma/G)$ be given by $f \mapsto \phi_f$ where
$$\phi_f(g) := f|_{[g]_k}(i).$$
(A priori ϕ_f is just a continuous function on G; we'll show that the map lands in L^2.)

By transitivity of the G-action on $\mathbb{H}$, ϕ is injective, and the map is clearly linear. Furthermore, ϕ_f satisfies:

(1) $\phi_f(\gamma g) = \phi_f(g)$ for all $\gamma \in \Gamma$, and thus descends to a function on $\Gamma \setminus G$.

(2) Let K be the stabilizer of i in G, namely

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =: r(\theta) \mid 0 \le \theta < 2\pi \right\}.$$

Then $\phi_f(g \cdot r(\theta)) = e^{-ik\theta}\phi(g)$ for all θ.

(3) $\phi_f(g)$ is bounded, and in particular is in $L^2(\Gamma \setminus G)$.

(4) ϕ_f is cuspidal; namely, for $g \in G$, a cusp s of Γ, σ sending ∞ to s, and h the period of $f|_{[\sigma]_k}$, we have

$$\int_0^1 \phi_f\left(\sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

(i), (ii) and (iv) are easy computation. (iii) for even k follows by defining $g(z) = \mathrm{Im}(z)^{k/2} f(z)$, which is Γ-invariant and thus descends to a function on $\Gamma \setminus \mathbb{H}^*$. But g is continuous and $\Gamma \setminus \mathbb{H}^*$ is compact, so g is bounded. But it is easy to see that the boundedness of g is equivalent to the boundedness of ϕ_f.

We now wish to extrinsically identify the image of $S_k(\Gamma)$ in $L^2(\Gamma \setminus G)$. We do this via the representation theory of the natural (right regular) representation R of G on $L^2(\Gamma \setminus G)$ given by

$$R(g)\phi(h) = \phi(hg).$$

In particular, we will find an operator on $L^2(\Gamma \setminus G)$ (namely, a Laplacian) which commutes with R; then its eigenspaces will give good finite-dimensional subspaces of $L^2(\Gamma \setminus G)$, out of which we'll be able to pick out our cusp forms.

There are three reasonable viewpoints on this Laplacian Δ; I'll proceed from the least computational to the most. First, the Killing form κ on $\mathfrak{sl}_2(\mathbb{R})$ is negative definite, so its additive inverse $-\kappa$ induces an invariant Riemannian metric on G which descends to $\Gamma \setminus G$ and whose Laplacian manifestly commutes with R. Alternately and equivalently (check this), we may take the basis

$$\ell_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ell_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ell_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{R})$; then the one-parameter subgroups in G corresponding to these Lie algebra elements satisfy $U_j(t) := R(\exp(t\ell_j)) = e^{-itH_j}$ for certain self-adjoint unbounded operators H_j on $L^2(\Gamma \setminus G)$. We may set

$$\Delta = -\frac{1}{4}(H_0^2 - H_1^2 - H_2^2).$$

Finally, setting

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\}$$

we may write $G = NAK$. The group NA acts simply transitively on the upper half-plane via

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} i = x + iy,$$

and $G = NAK$, so we may write each element of G as a triple (x, y, θ). In this parametrization the Laplacian satisfies

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}.$$

Let $A_k^2(\Gamma)$ be the subspace of $L^2(\Gamma \setminus G)$ satisfying $\phi(g \cdot r(\theta)) = \phi(g)e^{-ik\theta}$, $\Delta\phi = -\frac{k}{2}(\frac{k}{2} - 1)\phi$, and ϕ cuspidal. Then

Proposition 1. *The map $f \mapsto \phi_f(g)$ is an isomorphism between $S_k(\Gamma)$ and $A_k^2(\Gamma)$.*

Proof. We already know $f \mapsto \phi_f$ is injective; we must show it lands in $A_k^2(\Gamma)$ and is onto. In the (x, y, θ) parametrization above, $\phi_f(g) = y^{k/2} f(z) e^{-ik\theta}$, and a straightforward computation thus gives that

$$\Delta\phi_f(g) = -\frac{k}{2}(\frac{k}{2} - 1)\phi_f(g).$$

(This uses that f is holomorphic.) So if f is a cusp form, ϕ_f is in $A_k^2(\Gamma)$.

Now let $\phi \in A_k^2(\Gamma)$, and let $f(z) = \phi(g)j(g, i)^k$, where g satisfies $g(i) = z$. As $\phi(g \cdot r(\theta)) = phi(g)e^{-ik\theta}$, this is well-defined, and $\phi_f = \phi$ trivially. So we wish to check that f is a cusp form. We will do this part when we have a bit more technology. (IN SECTIONS 8 AND 9 OF GELBART, COME BACK TO THIS.) The content is showing that f is holomorphic. $\square$

This proposition motivates the definition of more general (not necessarily holomorphic) cusp forms. Namely, we make the following definition.

Definition 3 (Γ-automorphic form)**.** *A Γ-automorphic form ϕ on G is a smooth function satisfying*

1. *$\phi(\gamma g) = \phi(g)$ for all $\gamma \in \Gamma$;*
2. *ϕ is right K-finite, e.g. the span of $R(G)\phi(g)$ is finite dimensional;*
3. *ϕ is an eigenfunction of Δ; and*
4. *ϕ is slowly increasing, e.g. there are constants C and N such that*

$$|\phi(z, \theta)| \leq Cy^N$$

for large y.

If ϕ is in addition cuspidal *we call it a Γ-cusp form.*

Maass forms are an important example of such functions.

3. Beginning the Adelic Theory

We work over $\mathbb{Q}$, but essentially everything will carry over identically for number fields with class number one; I'll try to describe what happens with larger class number if I can. Let p be a place of $\mathbb{Q}$ (possibly infinite) and $\mathbb{Q}_p$ the completion; if p is finite let $\mathcal{O}_p$ be the ring of integers of $\mathbb{Q}_p$. Let $GL(2, \mathbb{Q}_p)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Q}_p$, topologized via the inclusion

$$GL(2, \mathbb{Q}_p) \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p) \times \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$$

$$g \mapsto (g, g^{-1})$$

or equivalently via the natural inclusion

$$GL(2, \mathbb{Q}_p) \to \mathrm{End}(\mathbb{Q}_p^2) = \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$$

where the latter is given the compact-open topology. For each p we take K_p to be the maximal compact (if p is finite, open) subgroup of G_p, namely $GL(2, \mathcal{O}_p)$ for finite p and $O(2, \mathbb{R})$ for $p = \infty$. As a set, $GL(2, \mathbb{A}_\mathbb{Q})$ is simply the $\mathbb{A}_\mathbb{Q}$-points of $GL(2)$, but again the topology is given by the inclusion

$$GL(2, \mathbb{A}_\mathbb{Q}) \to \mathrm{Mat}_{2\times 2}(\mathbb{A}_\mathbb{Q}) \times \mathrm{Mat}_{2\times 2}(\mathbb{A}_\mathbb{Q})$$

$$g \mapsto (g, g^{-1})$$

or equivalently

$$GL(2, \mathbb{A}_Q) \to \mathrm{End}(\mathbb{A}_\mathbb{Q}^2)$$

with the latter given the compact-open topology. This insures that the map $g \mapsto g^{-1}$ is continuous, and echoes the correct topology on the ideles (which are the center of $GL(2, \mathbb{A}_\mathbb{Q})$). Alternately, we may view $GL(2, \mathbb{A}_\mathbb{Q})$ as the restricted product over all places p of $\mathbb{Q}$ of the $GL(2, \mathbb{Q}_p)$, where the product is restricted over the K_p; indeed, the center of $GL(2, \mathbb{A}_\mathbb{Q})$ is the ideles of $\mathbb{Q}$, embedded as the scalar matrices. Note that $GL(2, \mathbb{Q})$ maps naturally into $GL(2, \mathbb{A}_\mathbb{Q})$.

For $GL(1, \mathbb{A}_\mathbb{Q})$ we have the following formula:

$$GL(1, \mathbb{A}_\mathbb{Q}) = \mathbb{Q}^\times \mathbb{R}_{>0}^\times \prod_{p \text{ finite}} \mathcal{O}_p^*,$$

and indeed an analogous decomposition holds for any number field with class number 1. An analogous result holds for $GL(2, \mathbb{A}_K)$ if K has class number one, and follows from the following hard fact:

Theorem 1. *Let S be a finite set of places of K, including all the archimedean places. Then $SL(2, K)$ is dense in $SL(2, \mathbb{A}_K^S)$.*

Proof. See Brian Conrad's notes, "Strong Approximation in Linear Algebraic Groups." (`http://math.stanford.edu/~conrad/248BPage/handouts/strongapprox.pdf`) $\square$

Corollary 1. *Let K be a number field with class number one, $G_{\mathbb{A}} = GL(2, \mathbb{A}_K)$, $G_K = GL(2, K)$, G_∞^+ be the connected component of the identity in*

$$\prod_{v \text{ infinite}} GL(2, K_v)$$

and K'_p an open subgroup of K_p satisfying $K'_p = K_p$ for almost all p and $\det : K'_p \to \mathcal{O}_p^\times$ surjective for every p. Then

$$G_{\mathbb{Q}} = G_{\mathbb{Q}} G_\infty^+ \prod_{p \text{ finite}} K'_p.$$

Proof. Let $g \in GL(2, K)$; by our formula for the ideles we may write $\det(g) = ru$ with $r \in \mathbb{K}^\times$ and $u \in (\prod_{v \text{ infinite}} K_v^\times)^0$. Writing

$$g = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} g_1 \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

we have $g_1 \in SL(2, \mathbb{A}_K)$. Thus there exists some $\gamma \in SL(2, \mathbb{Q})$ with $\gamma^{-1} g_1 \in \prod SL(2, \mathcal{O}_p)$, which completes the proof. $\square$

In particular, we may for $p|N$ set

$$K'_p = K_p^N := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \bmod N \right\}.$$

This definition is designed so that

$$G_{\mathbb{Q}} \cap G_\infty^+ \prod K_p^N = \Gamma_0(N). \qquad (*)$$

Now let ψ be a grossencharacter, i.e. a unitary character of the ideles. Note that any character of $(\mathbb{Z}/N\mathbb{Z})^*$ gives rise to such a character, e.g. by composing with $\mathcal{O}_p^* \to (\mathbb{Z}/N\mathbb{Z})^*$ for $p|N$.

If ψ arises this way, we may use ψ to define a natural character on K_p^N via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi_p(a).$$

Then given $f \in S_k(N, \psi)$ we may define a function ϕ_f on $G_{\mathbb{A}}$ by

$$\phi_f(g) = f|_{[g_\infty]_k}(i)\psi(k_0)$$

where

$$g = \gamma g_\infty k_0, \gamma \in G_{\mathbb{Q}}, g_\infty \in G_\infty^+, k_0 \in \prod_{p|N} K_p^N \cdot \prod_{p \nmid N} K_p.$$

This is well-defined by $(*)$.

Let $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \psi)$ be the Hilbert space of left $G_{\mathbb{Q}}$-invariant L^2 functions on $G_{\mathbb{A}}$ satisfying $\phi(gz) = \phi(g)\psi(z)$ for z in the ideles. Then the map $f \mapsto \phi_f$ gives an isomorphism between $S_k(N, \psi)$ and those functions in $L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}}, \psi)$ satifying

(1) $\phi(gk_0) = \phi(g)\psi(k_0)$ for $k_0 \in \prod K_p^N$;
(2) $\phi(gr(\theta)) = e^{-ik\theta}\phi(g)$
(3)

$$\Delta\phi|_{G_\infty^+} = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\phi|_{G_\infty^+};$$

(4) ϕ is slowly increasing. This means that for every $c > 0$ and compact $\omega \subset G_{\mathbb{A}}$, there exist constants C and N such that

$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \leq C|a|^N$$

for $g \in \omega$, a an idele with $|a| > c$.

(5) ϕ is cuspidal, i.e.

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

for almost all g.

More generally, we make the following definition:

Definition 4 (Automorphic form on $GL(2)$)**.** *An automorphic form on $GL(2)$ is a function ϕ on $G_{\mathbb{A}}$ which is left $G_{\mathbb{Q}}$-invariant and satisfies $\phi(gz) = \phi(g)\psi(z)$ for z in the ideles, such that*

(1) *ϕ is right K-finite, with*

$$K = \prod K_p,$$

(2) *ϕ is slowly increasing in the sense of (4) above, and*

(3) *as a function on G_∞, ϕ is smooth and the span of its orbit under $Z(U(Lie(G_\infty)))$ is finite-dimensional.*

If ϕ is cuspidal in the sense of (5) above, we say it is a cusp form.

4. Fourier Coefficients

The Fourier expansion of a ψ-cusp form admits a clean formulation in the adelic setting. Namely, for almost all $g \in G_{\mathbb{A}}$, we may define the function

$$\phi_g : \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}$$
$$x \mapsto \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$$

which is a square-integrable function on $\mathbb{Q} \setminus \mathbb{A}$.

Recall (from last quarter) that $\mathbb{Q} \setminus \mathbb{A}$ is Pontrjagin dual to $\mathbb{Q}$, as follows. Let $\chi_\infty : \mathbb{R} \to S^1$ be given by

$$\chi_\infty : x \mapsto e^{2\pi i x}$$

and $\chi_p : \mathbb{Q}_p \to S^1$ be the composition

$$\chi_p : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to S^1$$

where the latter map sends $1/p^k \mapsto \zeta_{p^k}$. (Here $\{\zeta_{p^k}\}_k$ is a collection of primitive p^k-th roots of unity satisfying $\zeta_{p^k}^p = \zeta_{p^{k-1}}$.) Then $\chi := \chi_\infty \cdot \prod_p \chi_p$ is a character on $\mathbb{A}$ vanishing on $\mathbb{Q}$, and so descends to a character of $\mathbb{Q} \setminus \mathbb{A}$. The isomorphism between $\mathbb{Q} \setminus \mathbb{A}$ and its group of characters $\mathbb{Q}$ is exhibited by

$$\xi \mapsto (x \mapsto \chi(\xi x))$$

for $\xi \in \mathbb{Q}$.

Thus by the usual theory of Pontrjagin duality, each ϕ_g may be written as

$$\phi_g(x) := \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\xi \in \mathbb{Q}} a_\xi(g) r(\xi x)$$

where

$$a_\xi(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \phi_g(x)\overline{r(\xi x)} dx.$$

Cuspidality is equivalent to the vanishing of $a_0(g)$ for almost all g, as before.

Let us compare this notion of q-expansion to the classical notion. Namely, let $f \in S_k(SL(2,\mathbb{Z}))$ be a cusp form, with fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

about the cusp at infinity. Then we claim that for $\phi = \phi_f$, $y > 0, x \in \mathbb{R}$ and $z = x + iy$,

$$\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right) = \sum_n a_n e^{2\pi i n z} = f(z).$$

In particular,

$$a_\xi\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right) = a_n e^{-2\pi n y}$$

for $\xi \in \mathbb{Z}$ and 0 otherwise.

Proof. It suffices to prove the latter statement. First we analyze the case where ξ is not an integer. Then there is some prime p appearing in the denominator of ξ, to order m. As ϕ_f is right-invariant under $GL(2, \mathcal{O}_p)$, we may write

$$\begin{aligned} a_\xi\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right) &= \int_{\mathbb{Q}\setminus\mathbb{Q}} \phi_f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} & p^{m-1}\\ 0 & 1\end{pmatrix}\right)\overline{r(\xi x)}dx \\ &= \int_{\mathbb{Q}\setminus\mathbb{A}} \phi_f\left(\begin{pmatrix}1 & x + p^{m-1} & \\ & 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\overline{r(\xi x)}dx \\ &= r(\xi p^{m-1}\phi_\xi)\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\end{aligned}$$

But $r(\xi p^{m-1}) \neq 1$, so

$$a_\xi\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right) = 0$$

as desired.

On the other hand, if $\xi \in \mathbb{Z}$, this is a straightforward computation. □

5. Hecke Operators

We'll work with $S_k(\Gamma_0(N))$; from what we've already done, we know that this corresponds to some subspace of

$$L^2(G_\mathbb{Q} \setminus G_\mathbb{A}/K_0^N)$$

with

$$K_0^N = \prod_{p|N} K_p^N \cdot \prod_{p\nmid N} K_p.$$

Let

$$H_p := K_p \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} K_p$$

for $p \nmid N$. For ϕ a function on $\mathbb{G}_\mathbb{A}$ right-invariant under the action by K_p, we let

$$\tilde{T}(p)\phi(g) = \int_{H_p} \phi(gh)dh$$

(this is right convolution with the characteristic function of H_p. One may check that the properties of $S_k(\Gamma_0(N))$ are preserved under this convolution.

We wish to understand the relationship between this notion of Hecke operators and the classical one. Namely,

Proposition 2. *If $\phi = \phi_f$ for $f \in S_k(\Gamma_0(N))$, we have*

$$p^{\frac{k}{2}-1}\tilde{T}(p)\phi = \phi_{T(p)f}.$$

Proof. Explicit computation gives

$$\begin{aligned}
p^{\frac{k}{2}-1}\tilde{T}(p)\phi_f(g) &= p^{\frac{k}{2}-1}\sum_{b=0}^{p-1}\phi_f\left(g\begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix}\right) + p^{\frac{k}{2}-1}\phi\left(g\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right) \\
&= p^{-1}\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1}f(pz) \\
&= p^{k-1}\sum_{a>0, ad=p}\sum_{b=0}^{d-1} f\left(\frac{az+d}{d}\right)d^{-k} \\
&= \phi_{T(p)f}(g)
\end{aligned}$$

as desired. (Here $z = g_\infty i$.) $\square$