KERNELS OF SURJECTIVE BUNDLE MAPS NEED NOT BE BUNDLES

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The purpose of this map is to exhibit a surjective map of free modules whose kernel is not locally free. Viewed geometrically, this provides a surjective map of trivial (infinite-dimensional) vector bundles whose kernel is not a vector bundle. Of course, the kernel is projective.

Let M be a projective module over a ring R which is not locally free–such things exist e.g. by section 15 in

`http://www.math.columbia.edu/algebraic_geometry/stacks-git/examples.pdf`;

to be explicit, there $R = \prod_{i \in \mathbb{N}} \mathbb{F}_2$ and M is the ideal

$$\langle e_n \mid e_n = (\underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots) \rangle.$$

Then M is a direct summand of a free module, say $M \oplus M' = R^{\oplus I}$.

Writing

$$R^{\oplus I'} = (M \oplus M') \oplus (M \oplus M') \oplus \cdots = M \oplus (M' \oplus M) \oplus (M' \oplus M) \oplus \cdots = M \oplus R^{\oplus J'}$$

gives M as a direct summand of a free module with free complement. Then there is a short exact sequence

$$0 \to M \to R^{\oplus I'} \to R^{\oplus J'} \to 0$$

exhibiting M as the kernel of a surjective map of free modules, where the map $R^{\oplus I'} \to R^{\oplus J'}$ is the projection.