ÉTALE COHOMOLOGY OF CURVES

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1. Introduction

The main theorem we will prove today is the following:

Theorem 1. *Let k be an algebraically closed field of characteristic prime to n. Let X/k be a smooth projective connected curve of genus g. Then*

$$H^0(X_{ét}, \mu_n) \simeq \mu_n(k) \ (\simeq \mathbb{Z}/n\mathbb{Z} \ \textit{noncanonically})$$
$$H^1(X_{ét}, \mu_n) \simeq \mathrm{Pic}(X)[n] \ (\simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \ \textit{noncanonically})$$
$$H^2(X_{ét}, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}.$$

This should be though of as an analogue to the well-known computation of the cohomology of curves over $\mathbb{C}$.

Theorem 2. *If X is a smooth projective curve over $\mathbb{C}$ of genus g, we have*

$$H^0(X(\mathbb{C})^{an}, \mathbb{Z}) \simeq \mathbb{Z}$$
$$H^1(X(\mathbb{C})^{an}, \mathbb{Z}) \simeq H^1(\mathrm{Jac}(X)(\mathbb{C})^{an}, \mathbb{Z}) \ (\simeq \mathbb{Z}^{2g} \ \textit{noncanonically})$$
$$H^2(X(\mathbb{C})^{an}, \mathbb{Z}) \simeq \mathbb{Z}.$$

Observe that there is a canonical generator of $H^2(X_{\text{ét}}, \mu_n)$—just as in the complex case, smooth projective varieties are canonically oriented.

Theorem 1 is the first indication we have that étale cohomology is a good theory, in that it agrees with our intuitions from complex geometry. This note aims to give an essentially complete proof of Theorem 1—the main difference from standard sources will be that I will use twisted sheaves instead of central simple algebras in the computation of the Brauer group of the function field of X.

Here is a brief outline of the steps in the proof of Theorem 1. Our main tools will be the following two exact sequences of sheaves on $X_{\text{ét}}$.

Lemma 1 (Kummer Exact Sequence)**.** *Let X be any scheme with n invertible in $\mathcal{O}_X$. Then there is an exact sequence of sheaves*

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \to 1$$

on $X_{ét}$.

Lemma 2 (Divisor Exact Sequence)**.** *Let X be an integral, separated, locally factorial scheme, and let η be the inclusion of the generic point. Then there is an exact sequence of sheaves*

$$1 \to \mathbb{G}_m \to \eta_* \mathbb{G}_m \to \mathrm{Div} \to 0$$

on $X_{ét}$. Here Div *is the sheaf*

$$\mathrm{Div} := \bigoplus_{\nu \in X^{codim\ 1}} \nu_* \mathbb{Z}.$$

The direct sum runs over all codimension 1 *points of X.*

This latter exact sequence is an étale version of the usual sequence relating regular functions, meromorphic functions, and divisors on a normal scheme. We will use Lemma 2 to compute the cohomology of $\mathbb{G}_m$, and deduce from this the cohomology of μ_n using Lemma 1.

In order to use Lemma 2, we have to understand the cohomology of $\eta_* \mathbb{G}_m$ and Div. We will show:

Lemma 3. *Let C be a curve over an separably closed field. Then* Div *is acyclic, i.e.*

$$H^i(C_{\acute{e}t}, \mathrm{Div}) = 0$$

for $i > 0$.

Lemma 4 (Hilbert 90)**.** *Let X be a scheme. Then $H^1(X_{\acute{e}t}, \mathbb{G}_m) = H^1(X_{Zar}, \mathbb{G}_m)$.*

Lemma 5. *Let $k(C)$ be the function field of a curve over an algebraically closed field k (e.g. a finitely generated field of transcendence degree 1 over k). Then*

$$H^2(k(C), \mathbb{G}_m) = 0.$$

Likewise for every Galois extension $k(C')/k(C)$,

$$H^2(k(C')/k(C), \mathbb{G}_m) = 0.$$

Lemma 5 will be by far the most involved part of the proof. It will also be the only part where I substantially depart from standard sources, since I prefer twisted sheaves to central simple algebras. Combining these last two lemmas with some results from group cohomology, we'll be able to conclude that

Corollary 1. *Let $k(C)$ be the function field of a curve over an algebraically closed field. Then*

$$H^i(k(C), \mathbb{G}_m) = 0$$

for all $i > 0$.

and

Corollary 2. *Let C be a smooth projective connected curve over an algebraically closed field, with η the inclusion of the generic point. Then*

$$H^i(C_{\acute{e}t}, \eta_* \mathbb{G}_m) = 0$$

for $i > 0$.

Using Lemma 2, this will imply that

Corollary 3. *Let C be a smooth projective connected curve over an algebraically closed field k. Then*

$$H^0(C, \mathbb{G}_m) = k^*,$$
$$H^1(C, \mathbb{G}_m) = \mathrm{Pic}(C), \textit{ and}$$
$$H^i(C, \mathbb{G}_m) = 0 \textit{ for } i > 1.$$

Finally, using Lemma 1 and the structure of $\mathrm{Pic}(C)$, we'll be able to deduce Theorem 1.

2. The Kummer and Divisor Exact Sequences

Let's begin by proving Lemmas 1 and 2.

2.1. The Kummer Exact Sequence.

Recall the statement of Lemma 1:

Lemma 1. *Let X be any scheme with n invertible in $\mathcal{O}_X$. Then there is an exact sequence of sheaves*

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \to 1$$

on $X_{\acute{e}t}$.

Proof. Recall that $\mathbb{G}_m$ is represented by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ and μ_n is represented by $\operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$. Zeb showed last time that representable functors are sheaves in the étale topology (this followed from étale descent), so both μ_n and $\mathbb{G}_m$ are indeed sheaves. It's easy to see that for any scheme U étale over X,

$$0 \to \mu_n(U) \to \mathbb{G}_m(U) \xrightarrow{x \mapsto x^n} \mathbb{G}_m(U)$$

is exact, so the sequence in question is left exact. To see right exactness, we must show that for any $f \in \mathbb{G}_m(U)$, there exists some étale cover $s : V \to U$ and some $g \in \mathbb{G}_m(V)$ with $g^n = s^*(f)$.

Consider a cartesian square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow f \\ (\mathbb{G}_m)_{\mathbb{Z}[\frac{1}{n}]} & \xrightarrow{t\mapsto t^n} & (\mathbb{G}_m)_{\mathbb{Z}[\frac{1}{n}]} \end{array}$$

The map $V \to U$ is étale because it is the base-changed from of an étale map; the map $V \to (\mathbb{G}_m)_{\mathbb{Z}[\frac{1}{n}]}$ gives desired root of f. (In fact

$$V = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[t]/(t^n - f)),$$

whence t is the desired root of f; we deduce the étaleness of the map $V \to U$ from the étaleness of the n-th power map $\mathbb{G}_m \xrightarrow{x\mapsto x^n} \mathbb{G}_m$ over $\mathbb{Z}[\frac{1}{n}]$.) If you think a bit about this argument, you'll see that we've shown that if $X \to Y$ is a surjective étale map, the map of étale sheaves $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$ is an epimorphism (in the category of étale sheaves). □

2.2. **Pushforwards of Étale Sheaves.** Before explaining the statement of Lemma 2 (and proving it) we must first discuss pushforwards of étale sheaves. Let us first briefly consider the pushforward of sheaves on topological spaces.

Suppose $p : X \to Y$ is a continuous map of spaces, and $\mathcal{F}$ is a presheaf on X. We may view $\mathcal{F}$ as a functor $\mathcal{F} : \mathrm{Opens}(X)^{op} \to \mathcal{C}$, where $\mathrm{Opens}(X)$ is the category whose objects are open sets in X, with a unique morphism $U \to V$ if $U \subset V$. Then $p : X \to Y$ induces a functor $p^{-1} : \mathrm{Opens}(Y) \to \mathrm{Opens}(X)$, sending an open set in Y to its preimage under p. The pushforward $p^*\mathcal{F}$ is defined via $p_*\mathcal{F} = \mathcal{F} \circ p^{-1}$. If $\mathcal{F}$ was a sheaf, so is $p_*\mathcal{F}$.

Now, suppose $p : X \to Y$ is a map of schemes. Such a map induces a "morphism of sites" $p^{-1} : Y_{\text{ét}} \to X_{\text{ét}}$—that is, a functor sending covers to covers—via

$$p^{-1}(U) := U \times_Y X, p^{-1}(f : U \to V) := (f \times_Y X : U \times_Y X \to V \times_Y X).$$

Again, for a presheaf $\mathcal{F}$ on $X_{\text{ét}}$ (e.g. a functor $\mathcal{F} : X_{\text{ét}}^{op} \to \mathcal{C}$) we define $p_*\mathcal{F} := \mathcal{F} \circ p^{-1}$. Again, we may check that if $\mathcal{F}$ was a sheaf, $p_*\mathcal{F}$ is as well (exercise).

The usual formalism of derived functors, with which I will assume the reader is familiar, goes through to obtain functors $R^i p_*$ as in the case of sheaves on a topological space.

2.3. **The Divisor Exact Sequence.** Now let's recall the statement of Lemma 2:

Lemma 2. *Let X be a Noetherian integral, separated, locally factorial scheme, and let η be the inclusion of the generic point. Then there is an exact sequence of sheaves*

$$0 \to \mathbb{G}_m \to \eta_*\mathbb{G}_m \to \mathrm{Div} \to 0$$

on $X_{ét}$. Here Div *is the sheaf*

$$\mathrm{Div} := \bigoplus_{\nu \in X^{codim\ 1}} \nu_*\mathbb{Z}.$$

The direct sum runs over all codimension 1 *points of X.*

Proof. Suppose $U \to X$ is étale. Then $\eta_*\mathbb{G}_m(U) = \Gamma(U \times_X \mathrm{Spec}(k(X)), \mathbb{G}_m)$. We claim that this is $\mathcal{M}^*(U)$ (that is, meromorphic functions on U). WLOG U is integral and affine, say $U = \mathrm{Spec}(A)$. $U \times_X \mathrm{Spec}(k(X))$ is étale over $\mathrm{Spec}(k(X))$; hence $U \times_X \mathrm{Spec}(k(X)) = \mathrm{Spec}(B)$, where B is a direct sum of fields seperable over $k(X)$. $A \subset B$, so B contains $\mathrm{Frac}(A)$; on the other hand, B is generated as a ring by $k(X)$ and A, so $B = \mathrm{Frac}(A)$. But $\mathbb{G}_m(\mathrm{Spec}(B)) = B^* = \mathcal{M}^*(U)$ as desired.

That $\mathrm{Div}(U)$ agrees with the usual Zariski sheaf of Weil divisors is immediate from the definition. Then exactness follows by the usual exactness of this sequence in the Zariski topology (e.g. Hartshorne 6.11). (Note that we need that an étale cover of a locally factorial scheme is locally factorial.) □

We may rather easily prove Lemma 3:

Lemma 3. *Let C be a curve over a separably closed field. Then* Div *is acyclic, e.g. $H^i(C_{ét}, \mathrm{Div}) = 0$ for $i > 0$.*

Proof. Let L be a separably closed, and let $\nu : \operatorname{Spec} L \to X$ be a morphism. As a first step, I claim that ν_* is exact. Indeed, any sheaf on $(\operatorname{Spec} L)_{\text{ét}}$ is constant (exercise), so it suffices to show that if $0 \to A \to B \to C \to 0$ is an exact sequence of Abelian groups, $0 \to \nu_*\underline{A} \to \nu_*\underline{B} \to \nu_*\underline{C} \to 0$ is exact. Let U/X be étale, so that $\nu_* U = (\operatorname{Spec} L)^r$ for some r. Then we have that

$$0 \to \nu_*\underline{A}(U) \to \nu_*\underline{B}(U) \to \nu_*\underline{C}(U) \to 0$$

is just

$$0 \to A^r \to B^r \to C^r \to 0$$

which is exact as desired. Thus the Leray spectral sequence gives that $H^i(X_{\text{ét}}, \nu_*\mathcal{F}) = H^i((\operatorname{Spec} L)_{\text{ét}}, \mathcal{F})$.

Now let $\nu : \operatorname{Spec} L \hookrightarrow C$ be the embedding of a closed point of C. We have by the above that $H^i(C_{\text{ét}}, \nu_*\mathbb{Z}) = H^i((\operatorname{Spec} L)_{\text{ét}}, \mathbb{Z})$. Recall that

$$\operatorname{Div} := \bigoplus_{\nu \in X^{\text{codim } 1}} \nu_*\mathbb{Z};$$

as cohomology commutes with direct sums, it thus suffices to show that $H^i((\operatorname{Spec} L)_{\text{ét}}, \mathbb{Z}) = 0$ for $i > 0$. But all sheaves on $(\operatorname{Spec} L)_{\text{ét}}$ are constant, so Γ is exact—hence all of its derived functors are zero. □

2.4. **Hilbert's Theorem 90.** Let's briefly recall the statement and proof of Hilbert's Theorem 90 (Lemma 4), and then see what we can conclude thus far about the cohomology of curves. Zeb proved Hilbert's Thoerem 90 last week, but the proof is so easy (after developing the theory of descent) that I can't resist recapitulating it.

Lemma 4. *Let X be a scheme. Then $H^1(X_{ét}, \mathbb{G}_m) = H^1(X_{Zar}, \mathbb{G}_m)$.*

Proof. A Čech cocycle for $H^1(X_{\text{ét}}, \mathbb{G}_m)$ is represented by an étale cover $U \to X$, and element $f \in \Gamma(U \times_X U, \mathbb{G}_m)$ so that if $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$ are the three projections, we have $\pi_{23}^* f \cdot \pi_{12}^* f = \pi_{13}^* f$. But this is precisely the same as descent data for a line bundle; by effectivity of étale descent of vector bundles, there is a Zariski Čech cocycle representing the same class. Thus the theorem is proven. □

We immediately deduce that if C is a smooth projective connected curve over a separably closed field, $H^1(C_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(C)$. This fact and the Kummer exact sequence gives that

$$H^1(C_{\text{ét}}, \mu_n) = \operatorname{Pic}(C)[n]$$

as desired (where we also use that $\mathbb{G}_m(C) \xrightarrow{t \mapsto t^n} \mathbb{G}_m(C)$ is surjective). Furthermore, we also may deduce from Lemma 3 that $H^i(C_{\text{ét}}, \mathbb{G}_m) = H^i(C_{\text{ét}}, \eta_*\mathbb{G}_m)$ for all $i > 0$. We will show that $H^i(C_{\text{ét}}, \eta_*\mathbb{G}_m) = 0$ for all $i > 0$; this is by far the most involved portion of the proof.

Remark 1. *Observe that we've also shown that if L/K is a Galois extension of fields,*

$$H^1(\operatorname{Gal}(L/K), L^*) = 0.$$

3. The Brauer Group

We were able to understand $H^1(X_{\text{ét}}, \mathbb{G}_m)$ via a geometric construction—that is, the main idea in the proof of Hilbert's Theorem 90 was to associate a line bundle to each cohomology class. Similarly, we will attempt to understand $H^2(X_{\text{ét}}, \mathbb{G}_m)$ via a geometric construction. This construction is not perfect in that

(1) For arbitrary schemes X, not every element of $H^2(X_{\text{ét}}, \mathbb{G}_m)$ will come from such a geometric object, and
(2) Each element of $H^2(X_{\text{ét}}, \mathbb{G}_m)$ may correspond to many geometric objects.

Consider the sheaf PGL_n on $X_{\text{ét}}$ which is the sheafification of the presheaf $U \mapsto \Gamma(U, GL_n)/\Gamma(U, \mathbb{G}_m)$. This sheaf is representable by a scheme, but we will not need this fact—we will use that PGL_{n+1} is the automorphism functor of $\mathbb{P}^n$ (this isn't so hard over a field, but is non-trivial in general). The short exact sequence

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

gives a long exact sequence of pointed sets

$$\cdots \to H^1(-, \mathbb{G}_m) \to H^1(-, GL_n) \to H^1(-, PGL_n) \xrightarrow{\delta_n} H^2(-, \mathbb{G}_m) \qquad (*)$$

where H^1 with coefficients in a non-abelian group means torsors over that group (e.g. 1-Čech cocycles moduli coboundaries). We won't actually need any facts about non-abelian cohomology, so don't worry if this is unfamiliar to you.

Let's first describe an element of $H^1(X_{\text{ét}}, PGL_n)$ explicitly. We may view such a thing as being represented by an étale cover $U \to X$, as well as a section $g \in \Gamma(U \times_X U, PGL_n)$ satisfying a cocycle condition $\pi_{23}^* g \cdot \pi_{12}^* g = \pi_{13}^* g$ (as in the proof of Hilbert 90). As PGL_n is the automorphism functor for $\mathbb{P}^{n-1}$, we may view this as descend data on $\mathbb{P}^{n-1}_U$. This descent data is effective, because $\mathbb{P}^{n-1}_U$ is anti-canonically polarized. Thus, we obtain a scheme P over X, so that $P \times_X U \simeq \mathbb{P}^{n-1}_U$.

Definition 1. *A scheme P/X so that there exists an étale cover U of X with $P \times_X U \simeq \mathbb{P}^n_U$ is called a Severi-Brauer variety of dimension n over X.*

Thus, we may view elements of $H^1(X_{\text{ét}}, PGL_n)$ as being represented by Severi-Brauer varieties over X.

Remark 2. *There are three common avatars that are used to study $H^1(X_{ét}, PGL_n)$:*

(1) *Azumaya algebras (if X is the spectrum of a field, these are called central simple algebras),*
(2) *Severi-Brauer vareities, and*
(3) *Twisted vector bundles.*

Later I will explain what a twisted vector bundle is, and how to pass between these three avatars of PGL_n-cocycles.

Let's also work out the boundary map δ_n from the exact sequence above. Suppose $[g] \in H^1(X_{\text{ét}}, PGL_n)$ is a Severi-Brauer variety, represented by some cocycle $g \in \Gamma(U \times_X U, PGL_n)$. Possibly after passing to a refinement U' of U, we may lift g to a cochain in $g' \in \Gamma(U' \times_X U', GL_n)$. As g was a cocycle, we have that $\pi_{23}^* g' \cdot \pi_{12}^* g' = \lambda \pi_{13}^* g'$, for some $\lambda \in \Gamma(U' \times_X U' \times_X U', \mathbb{G}_m)$. It is not too hard to check that λ is a 2-cocycle for $\mathbb{G}_m$; a diagram chase shows that

$$\delta_n([g]) = [\lambda].$$

Definition 2 (Brauer group)**.** *The Brauer group of X, denoted $Br(X)$, is the union of the images of $\delta_n : H^1(X_{ét}, PGL_n) \to H^2(X_{ét}, \mathbb{G}_m)$, that is, it is the collection of elements of $H^2(X_{ét}, \mathbb{G}_m)$ represented by Severi-Brauer varieties.*

3.1. **Twisted Sheaves.** It is not obvious from the definition that this is indeed a group—we will see this soon. One of the most useful modern techniques in the study of the Brauer group comes from an intermediate construction in our computation of δ_n above.

Definition 3 (Twisted Sheaf)**.** *Let $\alpha \in H^2(X_{ét}, \mathbb{G}_m)$ be a cohomology class. An α-twisted sheaf is "descent data for a quasi-coherent sheaf," twisted by α. Namely, let $U \to X$ be an étale cover and as usual, $\pi_i : U \times_X U \to U$ the projections, and similarly with $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$. An α-twisted sheaf is the data of a quasi-coherent sheaf $\mathcal{E}$ on U, and an isomorphism $\phi : \pi_1^* \mathcal{E} \xrightarrow{\sim} \pi_2^* \mathcal{E}$, so that $\pi_{23}^* \phi \circ \pi_{12}^* \phi = \lambda \pi_{13}^* \phi$, where $[\lambda] = \alpha \in H^2(X_{ét}, \mathbb{G}_m)$. Observe that if $\mathcal{E}$ is a vector bundle, we may (possibly after refining U to trivialize $\mathcal{E}$) view this descent data as the data of a section $g' \in \Gamma(U \times_X U, GL_n)$, as in our computation of δ_n; we call $\mathcal{E}$ an α-twisted vector bundle.*

The collection of α-twisted sheaves may be turned into a category $\mathrm{Coh}(X, \alpha)$ as follows: A morphism $(U, \mathcal{E}, \phi) \to (U', \mathcal{E}', \phi')$ is defined to be a morphism $f : r_1^*(\mathcal{E}, \phi) \to r_2^*(\mathcal{E}', \phi')$,.where $r_1 : U'' \to U, r_2 : U'' \to U'$ are the two étale covers. Two morphisms defined on U'', V'' are the same if they agree on a common refinement of U'', V''.

Observe that if $\mathcal{E}$ is an α-twisted vector bundle, the twisted descent data becomes honest descent data upon replacing $\mathcal{E}$ with $\mathbb{P}\mathcal{E}$. Similarly, if P is a Severi-Brauer variety, trivialized on some étale cover U of X, lifting the descent data for P from $\Gamma(U \times_X U, PGL_n)$ to $\Gamma(U' \times_X U', GL_n)$ (for some refinement U' of U) gives α-twisted descent data.

That said, the choice involved in the lift means that there is not a bijection between Severi-Brauer varieties representing α and α-twisted vector bundles—rather, examining the long exact sequence $(*)$ shows that there is a bijection

$$\{\text{Isomorphism classes of Severi-Brauer varieties}\} \xrightarrow{\sim} \{\text{Isomorphism classes of } \alpha\text{-twisted v.b.'s}\}/\operatorname{Pic}(X).$$

Proposition 1. *Let $\alpha, \alpha' \in H^2(X_{ét}, \mathbb{G}_m)$ be cohomology classes.*

(1) *α is a Brauer class if and only if there exists an α-twisted vector bundle.*
(2) *$\mathrm{QCoh}(X, \alpha)$ is a Grothendieck Abelian category (in particular, there are enough injectives).*
(3) *There are natural functors*

$$- \otimes - : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha + \alpha')$$

and

$$\mathrm{Hom}(-, -) : \mathrm{QCoh}(X, \alpha)^{op} \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha' - \alpha)$$

given by $\otimes$ and Hom *on twisted descent data.*
(4) *Similarly, $\bigwedge^n$ and Sym^n extend to functors $\mathrm{QCoh}(X, \alpha) \to \mathrm{QCoh}(X, n\alpha)$.*
(5) *$\mathrm{QCoh}(X, 0)$ is the usual category of quasi-coherent sheaves on X.*

Proof. While this proposition may seem intimidating, it is a (good) exercise in unwinding definitions. The only difficult part is in (2), showing that there is a set of generators—if X is locally Noetherian, one may imitate EGA I, 9.4.9. We will only use these results where X is the spectrum of a field, where everything is easy. So I will leave the proof in that case as a (mandatory) exercise. □

Corollary 4. *The Brauer group inside of $H^2(X_{ét}, \mathbb{G}_m)$ is indeed a group.*

Proof. This follows immediately from (3) above. A class $\alpha \in H^2(X_{ét}, \mathbb{G}_m)$ is in the Brauer group iff there is an α-twisted vector bundle. Then tensor product gives addition in the Brauer group, and $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$ gives inversion. □

Let's get our hands dirty with this concept a bit.

Proposition 2. *There is an α-twisted line bundle on X if and only if $\alpha = 0 \in H^2(X_{ét}, \mathbb{G}_m)$.*

Proof. If $\alpha = 0$, $\mathcal{O}_X$ is an α-twisted line bundle.

On the other hand, suppose $(U, \mathcal{L}, \phi)$ is an α-twisted line bundle. We may choose a cover $r : U' \to U$ so that $r^*\mathcal{L}$ is trivial; after choosing a trivialization, we may view $r^*\phi$ as an element of $\Gamma(U' \times_X U', \mathbb{G}_m)$, e.g. a 1-cochain for $\mathbb{G}_m$. But $d(r^*\phi)$ is a 2-cochain representing α by the definition of an α-twisted sheaf, so α is a coboundary. Thus $\alpha = 0$ as a cohomology class. □

Corollary 5. *Suppose $\mathcal{E}$ is an α-twisted vector bundle of rank n. Then α is n-torsion in $Br(X)$.*

Proof. $\bigwedge^n \mathcal{E}$ is an $n\alpha$-twisted line bundle—thus $n\alpha = 0$ in $Br(X)$. □

Twisted vector bundles have many of the same properties of vector bundles.

Proposition 3. *Suppose X is an affine scheme, and α a Brauer class on X. Then all short exact sequences of α-twisted vector bundles on X split.*

Proof. Suppose

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is a short exact sequence of α-twisted vector bundles. We wish to show that $\mathrm{Ext}^1_{\mathrm{QCoh}(X,\alpha)}(\mathcal{E}_3, \mathcal{E}_1) = 0$. But we have

$$\mathrm{Ext}^1_{\mathrm{QCoh}(X,\alpha)}(\mathcal{E}_3, \mathcal{E}_1) = H^1(X_{ét}, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) = 0$$

where we use that X is affine and that étale cohomology of coherent sheaves is the same as Zariski cohomology. □

Corollary 6. *Let $\mathcal{E}$ be an α-twisted vector bundle over the spectrum of a field. Then $\mathcal{E}$ is simple if and only if* $\mathrm{End}(\mathcal{E})$ *is a division algebra.*

Proof. Suppose $\mathcal{E}$ is simple. Then any non-zero endomorphism of $\mathcal{E}$ must have no kernel, as the kernel would be an α-twisted sub-bundle of $\mathcal{E}$. But we're over a field, so (working étale-locally), we see that an endomorphism with no kernel is an isomorphism.

On the other hand, if $\mathcal{E}$ is not simple, the proposition gives that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$; then projection to either factor is an a non-invertible endorphism. □

The classical study of the Brauer group was via central simple algebras, or (in the relative setting) Azumaya algebras, which are also classified by elements of $H^1(X_{\text{ét}}, PGL_n)$. If $\mathcal{E}$ is a twisted vector bundle corresponding to some class in $H^1(X_{\text{ét}}, PGL_n)$, the corresponding Azumaya algebra is $\text{End}(\mathcal{E})$ (exercise for those who know about Azumaya algebras).

Now we will see that the Brauer group is a useful tool in the study of $H^2(X_{\text{ét}}, \mathbb{G}_m)$. We have already seen that the former is a torsion subgroup of the latter—it is a deep theorem that in great generality—namely, for any scheme X with an ample invertible sheaf (in the sense of EGA II, 4.5.3 – any quasi-projective variety satisfies this condition),

$$Br(X) = H^2(X_{\text{ét}}, \mathbb{G}_m)_{tors}.$$

We will prove this for fields, following Serre's Local Fields, p. 159:

Proposition 4. *Let k be a field. Then $Br(\text{Spec}(k)) = H^2(\text{Spec}(k)_{ét}, \mathbb{G}_m)$; even stronger, if L/k is a Galois extension of degree n, the map $\delta_n : H^1(L/k, PGL_n) \to H^2(L/k, L^*)$ is surjective.*

Proof. Let α be an element of $H^2(\text{Spec}(k)_{\text{ét}}, \mathbb{G}_m)$. Let L/k be a Galois extension with Galois group G such that α is represented by a cocycle $a : G \times G \to L^*$, i.e. $a \in \Gamma(\text{Spec}(L \otimes_k L \otimes_k L), \mathbb{G}_m)$. Let $n = |G|$ and $V := \oplus_{g \in G} L[e_g]$. It will suffice to find $n \times n$ matrices

$$p_g \in GL(V)$$

for each $g \in G$ so that

$$a(s,t) = p_s \cdot s(p_t) \cdot p_{st}^{-1}$$

for this will show that

$$\delta_n : H^1(L/k, PGL_n) \to H^2(L/k, \mathbb{G}_m)$$

is surjective. These $p_g \in \Gamma(\text{Spec}(L \otimes_k L), GL(V))$ define an α-twisted vector bundle on $\text{Spec}(k)$.

We let $p_g(e_s) = a(g,s)e_{gs}$. Now we have that

$$a(s,t)p_{st}(e_u) = a(s,t)a(st,u)e_{stu}$$

$$p_s \cdot s(p_t)(e_u) = p_s \cdot s(a(t,u))e_{tu} = a(s,tu)s(a(t,u))e_{stu}.$$

But the cocycle condition says exactly that

$$s(a(t,u)) \cdot a(st,u)^{-1}a(s,tu)a(s,t)^{-1} = 1$$

which completes the proof. □

This proof seems pretty magical to me—historically, it is motivated by the "crossed product" central simple algebra associated to a 2-cocycle $a : G \times G \to L^*$. This algebra is the (opposite of the) endomorphism algebra of the twisted vector bundle (V, p_g). If anyone reading this knows of a more conceptual proof, I'd be thrilled to hear it.

4. The Brauer Group and Cohomology of $k(C)$

4.1. **The Brauer Group is Trivial.** We will now prove Lemma 5.

Lemma 5. *Let $k(C)$ be the function field of a curve over an algebraically closed field k (e.g. a finitely generated field of transcendence degree 1 over k). Then*

$$H^2(k(C), \mathbb{G}_m) = 0.$$

Likewise for every Galois extension $k(C')/k(C)$,

$$H^2(k(C')/k(C), \mathbb{G}_m) = 0.$$

By Proposition 4 it will suffice to show that the Brauer group of $k(C)$ is trivial. Let α be a Brauer class, and $\mathcal{E}$ an α-twisted sheaf of rank $n > 1$; it will suffice to show that $\mathcal{E}$ is not simple. By Corollary 6, we wish to show that $\text{End}(\mathcal{E})$ is not a division algebra. Let us turn this into a problem of algebraic geometry.

Definition 4 (Reduced Norm)**.** *Let X be a scheme and $\mathcal{E}$ an α-twisted vector bundle on X of rank n. The natural map $\det_{\mathcal{E}} : \text{End}(\mathcal{E}) \to \text{End}(\bigwedge^n \mathcal{E}) \simeq \mathcal{O}_X$ is called the reduced norm.*

An (étale) local computation shows that $s \in \mathrm{End}(\mathcal{E})$ is invertible if and only if $\det_{\mathcal{E}}(s)$ is a unit. Note that if we choose a local trivialization of $\mathrm{End}(\mathcal{E})$ as a vector bundle on X, we obtain that $\det_{\mathcal{E}}$ is a homogeneous polynomial of degree n in n^2 variables.

To see that $\mathrm{End}(\mathcal{E})$ is not a division algebra, it will suffice to show that there exists non-zero $s \in \mathrm{End}(\mathcal{E})$ so that $\det_{\mathcal{E}}(s) = 0$. We will in fact show something much stronger.

Theorem 3 (Tsen's Theorem)**.** *Let $k(C)$ be the function field of a curve C over an algebraically closed field k. Then if f is a homogeneous polynomial in n variables of degree d, with $d < n$, f has a non-trivial zero in $k(C)$.*

Proof. WLOG we may assume C is smooth and projective, say of genus g. Let X be a point on C, and consider the map

$$\Gamma(C, \mathcal{O}(rX))^n \xrightarrow{f} \Gamma(C, \mathcal{O}(rdX))$$

induced by f. By Riemann-Roch, for $r \gg 0$ we have that

$$\dim \Gamma(C, \mathcal{O}(rX)) = r - g + 1$$

and

$$\dim \Gamma(C, \mathcal{O}(rdX)) = rd - g + 1.$$

Thus we may view the map induced by f as a map

$$\mathbb{A}^{n(r-g+1)} \to \mathbb{A}^{rd-g+1}.$$

Each non-empty fiber must have dimension at least $n(r-g+1) - (rd-g+1)$, which is greater than zero for r sufficiently large. But the fiber over 0 is non-empty—as f is a homogeneous polynomial, it contains 0. So in particular, there is another point x in the fiber over zero for r sufficiently large, using that k is algebraically closed.

But then $x \in \Gamma(C, \mathcal{O}(rX))^n \subset k(C)^n$ is a non-trivial solution to f in $k(C)$. □

Remark 3. *An identical argument (replacing Riemann-Roch for curves with asymptotic Riemann-Roch), will show that if $k(X)$ is the function field of a projective variety X of dimension r over an algebraically closed field k, then any homogeneous polynomial f in n variables of degree d has a solution in $k(X)$ if $n > d^r$.*

We may now prove the lemma:

Proof of Lemma 5. Let $\alpha \in H^2(k(C), \mathbb{G}_m)$ be a cohomology class—we wish to show $\alpha = 0$. It will suffice to show that there is an α-twisted line bundle, by Proposition 2. By Proposition Proposition 1(1) and Proposition 4, there is an α-twisted vector bundle $\mathcal{E}$ of minimal rank. WLOG $n := \mathrm{rk}\,\mathcal{E} > 1$. It suffices to show that $\mathcal{E}$ is not simple, i.e. that $\mathrm{End}(\mathcal{E})$ is not a division algebra (by Corollary 6). But we shall see that $\det_{\mathcal{E}} : \mathrm{End}(\mathcal{E}) \to k(C)$ has a nontrivial zero.

Indeed $\det_{\mathcal{E}}$ is a form of degree n in n^2 variables—thus by Tsen's theorem, it has a non-trivial zero. □

Remark 4. *Observe that we've also shown that if $k(C')/k(C)$ is a Galois extension,*

$$H^2(\mathrm{Gal}(k(C')/k(C), k(C')^*) = 0.$$

4.2. **The Higher Cohomology of $k(C)$.** We now deduce that $H^i(k(C), \mathbb{G}_m) = 0$ for all $i > 0$. We will use the following theorem of Tate.

Theorem 4 (Tate)**.** *Let G be a finite group, and A a G-module. Suppose we have that for all subgroups $V \subset G$,*

$$H^i(V, A) = 0$$

for $i = 1, 2$. Then

$$H^i_T(G, A) = 0$$

for all $i \in \mathbb{Z}$. (Here H^i_T refers to Tate cohomology.)

Proof. For cyclic G, this is immediate from the 2-periodicity of cohomology.

For solvable G, we proceed by induction on $|G|$. Let $V \subset G$ be a subgroup so that G/V is cyclic. Then the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H_T^p(G/V, H^q(V, A)) \implies H_T^{p+q}(G, A)$$

gives the result immediately, by the induction hypothesis and the cyclic case.

Now for arbitrary G, let G_p be a p-Sylow subgroup of G; we may assume $H_T^i(G_p, A) = 0$ by the solvable case. Then the composition

$$H_T^i(G, A) \xrightarrow{\text{res}} H_T^i(G_p, A) \xrightarrow{\text{cor}} H_T^i(G, A)$$

is multiplication by $[G : G_p]$ which is prime to p, so the map on p-power torsion

$$H_T^i(G, A)[p^\infty] \xrightarrow{\text{res}} H_T^i(G_p, A)[p^\infty]$$

is injective. Thus $H_T^i(G, A)$ has no torsion. But G is finite, so by Maschke's theorem group cohomology is torsion, so we have $H_T^i(G, A) = 0$ as desired. □

Corollary 7. *Let G be finite and A a G-module. Suppose we have that for all subgroups $V \subset G$,*

$$H^i(V, A) = 0$$

for $i = 1, 2$. Then

$$H^i(G, A) = 0$$

for all $i > 0$.

Proof. $H_T^i(G, A) = H^i(G, A)$ for $i > 0$. □

We may now prove Corollary 1.

Corollary 1. *Let $k(C)$ be the function field of a curve over an algebraically closed field. Then*

$$H^i(k(C), \mathbb{G}_m) = 0$$

for all $i > 0$.

Proof. Let $k(C')/k(C)$ be a Galois extension of function fields of curves over algebraically closed fields; then $H^i(\text{Gal}(k(C')/k(C)), k(C')^*) = 0$ for $i = 1, 2$. Any subgroup $V \subset \text{Gal}(k(C')/k(C))$ is likewise the Galois group of an extension of function fields of curves (namely $k(C')/k(C')^V$), so by Lemmas 4 and 5 we likewise have that $H^i(V, k(C')^*) = 0$ for $i = 1, 2$. Thus by Corollary 7,

$$H^i(k(C')/k(C), k(C')^*) = 0$$

for all $i > 0$. Taking limits gives the claim. □

4.3. **The Cohomology of $\eta_*\mathbb{G}_m$.** We now will deduce Corollary 2:

Corollary 2. *Let C be a smooth projective connected curve over an algebraically closed field, with η the inclusion of the generic point. Then*

$$H^i(C_{\acute{e}t}, \eta_*\mathbb{G}_m) = 0$$

for $i > 0$.

Proof. It will suffice to show that $R^i\eta_*\mathbb{G}_m = 0$ for $i > 0$, because if this is the case, the Leray spectral sequence will imply that $H^i(C_{\text{ét}}, \eta_*\mathbb{G}_m) = H^i(k(C), \mathbb{G}_m) = 0$ for $i > 0$. But $R^i\eta_*\mathbb{G}_m$ is the sheafification of the presheaf

$$U \mapsto H^i(U \times_C \text{Spec}(k(C)), \mathbb{G}_m)$$

which is zero by Corollary 1, since each $U \times_C \text{Spec}(k(C))$ is a disjoint union of spectra of function fields of curves. □

5. Completing the Proof

We may finally deduce Corollary 3:

Corollary 3. *Let C be a smooth projective connected curve over an algebraically closed field k. Then*

$$H^0(C, \mathbb{G}_m) = k^*,$$
$$H^1(C, \mathbb{G}_m) = \mathrm{Pic}(C), \text{ and}$$
$$H^i(C, \mathbb{G}_m) = 0 \text{ for } i > 1.$$

Proof. That $H^0(C_{\text{ét}}, \mathbb{G}_m) = k^*$ is immediate from the definition; $H^1(C_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(C)$ by Lemma 4. Thus we need only prove the vanishing statement. But this is immediate from the divisor exact sequence (Lemma 2) and the acyclicity of Div (Lemma 3) and $\eta_* \mathbb{G}_m$ (Corollary 2). □

We now state without proof the properties of $\mathrm{Pic}(C)$ which allow us to deduce Theorem 1.

Theorem 5 (Structure of $\mathrm{Pic}(C)$)**.** *Let C be a smooth, connected, projective curve of genus $g$over a field k, and suppose C has a rational point. Then there is a short exact sequence*

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(C) \to \mathbb{Z} \to 0.$$

Here $\mathrm{Pic}^0(C)$ is the k-points of an Abelian variety of dimension g—thus in particular, if k is algebraically closed it is divisible, and if as well $(n, \mathrm{char}(k)) = 1$, we have that $\mathrm{Pic}^0(C)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

We may finally conclude the proof of our main theorem:

Theorem 1. *Let k be an algebraically closed field of characteristic prime to n. Let X/k be a smooth projective connected curve of genus g. Then*

$$H^0(X_{ét}, \mu_n) \simeq \mu_n(k) \ (\simeq \mathbb{Z}/n\mathbb{Z} \text{ noncanonically})$$
$$H^1(X_{ét}, \mu_n) \simeq \mathrm{Pic}(X)[n] \ (\simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \text{ noncanonically})$$
$$H^2(X_{ét}, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}.$$

Proof. We use the Kummer exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 1.$$

As X is proper and k is algebraically closed, $H^0(X_{\text{ét}}, \mu_n) = \mu_n(k) \simeq \mathbb{Z}/n\mathbb{Z}$ non-canonically. $\mathbb{G}_m(X) \to \mathbb{G}_m(X)$ is surjective (again as X is proper and k is algebraically closed, this is the n-th power map $k^* \to k^*$). So

$$H^1(X_{\text{ét}}, \mu_n) = \ker(H^1(X_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\cdot n} H^1(X_{\text{ét}}, \mathbb{G}_m),$$

which is $\mathrm{Pic}(X)[n]$ by definition.

Finally, as $H^i(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $i > 1$ by Corollary 3, we have

$$H^2(X_{\text{ét}}, \mu_n) = \mathrm{coker}(\mathrm{Pic}(X) \xrightarrow{\cdot n} \mathrm{Pic}(X))$$

and

$$H^i(X_{\text{ét}}, \mu_n) = 0$$

for $i > 2$. But $\mathrm{Pic}^0(X)$ is divisible, so $H^2(X_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$ as desired. □