CURVES HAVE FINITE AUTOMORPHISM GROUP

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Let X be a smooth projective curve of genus $g \geq 2$, over a field k.

Proposition 1. *The automorphism group of X is finite.*

In fact, a slight modification of the argument will give:

Theorem 1. *Let X be a smooth projective variety, with $\dim(X) < \mathrm{char}(k)$. Suppose ω_X is ample and X lifts to $W_2(k)$. Then the automorphism group of X is finite.*

Proof of Proposition. WLOG $k = \bar{k}$. Choose n so that $\omega_X^{\otimes n}$ is very ample, and let

$$\iota : X \hookrightarrow \mathbb{P}\Gamma(X, \omega_X^{\otimes n})$$

be the natural closed embedding. The action of $\mathrm{Aut}(X)$ (which we view as an abstract group—this proof will not use the representability of the automorphism functor) extends to an action on $\mathbb{P}\Gamma(X, \omega_X^{\otimes n})$ via pullback of differentials; ι is equivariant for this action. As the action of $\mathrm{Aut}(X)$ on X is faithful, we obtain an injection

$$\mathrm{Aut}(X) \hookrightarrow PGL(\Gamma(X, \omega_X^{\otimes n})).$$

Let G be the closure of the image of $\mathrm{Aut}(X)$ in $PGL(\Gamma(X, \omega_X^{\otimes n}))$. Suppose for the sake of contradiction that $\mathrm{Aut}(X)$ is infinite; then G has positive dimension (and also acts on X). Consider non-zero $v \in T_eG$. We claim that the vector field on X induced by v is non-zero (in characteristic zero this is clear, but in characteristic p an argument is required—see the remark at the bottom of the page).

Indeed, suppose $v \in G(k[\epsilon]/\epsilon^2)$ acted trivially on $X_{k[\epsilon]/\epsilon^2}$. Then the action on $\mathbb{P}\Gamma(X, \omega_X^{\otimes n})_{k[\epsilon]/\epsilon^2}$ (induced by pullback) would also be trivial—but it's easy to see that any non-zero tangent vector in $T_ePGL(V)$ acts non-trivially on $\mathbb{P}V_{k[\epsilon]/\epsilon^2}$.

So we've obtained a non-zero element $v \in H^0(X, T_X)$. But T_X has negative degree by the assumption on the genus, so $H^0(X, T_X) = 0$; this is the desired contradiction. $\square$

The only case we used that X was a curve was in the very last step, where we show $H^0(X, T_X) = 0$.

Proof of Theorem. We proceed identically as in the previous proof; we must only show that $H^0(X, T_X) = 0$. But

$$H^0(X, T_X) \simeq H^{\dim(X)}(X, \Omega_X^1 \otimes \omega_X)^\vee$$

by Serre duality. And if $\mathrm{char}(k) > \dim(X)$ and X lifts to $W_2(k)$, this latter group is zero by Kodaira vanishing. $\square$

Remark 1. *It's possible for a group to act faithfully on a variety with zero derivative—let k be a field of characteristic $p > 0$ and consider $\mathbb{G}_m$ acting on $\mathbb{A}^1$ via*

$$\mathbb{G}_m(R) \times \mathbb{A}^1(R) \to \mathbb{A}^1(R)$$
$$(\alpha, r) \mapsto \alpha^p r.$$

Remark 2. *I don't know if the theorem is true if one drops the condition on the characteristic of k. Certainly there are varieties with ample canonical bundle but with non-zero vector fields, so the argument won't work.*