<u>The</u> Derived Category of Coherent Sheaves on $\mathbb{P}^n$

Reminder: V is a vector space$/k$ of dim $n+1$

$\mathbb{P}V^* = \mathrm{Proj}(\mathrm{Sym}^\cdot V^*)$, $V = \Gamma(\mathbb{P}V, \mathcal{O}(1))$

Given a graded module $M^\cdot$ over $\mathrm{Sym}^\cdot V$:

Get (1) A sheaf $\mathcal{F}_M$ on $\mathbb{P}V$ (quasi-coherent — coherent if $M^\cdot$ is f.g.)

via $\hat{\mathcal{F}}_M(\mathbb{P}V^* \setminus V(f)) = (M^\cdot_f)_0$

(2) A sheaf on $\mathrm{Spec}(\mathrm{Sym}^\cdot V) \cong \mathbb{A}^{n+1}$ $(\hat{\mathcal{F}})$ w/ an equivariant $\mathbb{G}_m$ action, e.g. for

$t \in \mathbb{G}_m(T)$, $(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}(T)$, $f \in \hat{\mathcal{F}}(T)_n = \Gamma(T, t^* \hat{\mathcal{F}}_0)$

$t \cdot f = t^n f$ — equivariance means

$t \cdot ((x_0, \ldots, x_n) f) = (t \cdot (x_0, \ldots, x_n))(t \cdot f)$.

(This is the same as a graded module! $M_n := \{ f \mid t \cdot f = t^n f \}$)

To go back: $\hat{\mathcal{F}} \mapsto \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{P}V^*, \mathcal{F}(n))$

Geometric picture:

$$\begin{tikzcd} \mathbb{G}_m \circlearrowright \hat{\mathcal{F}} \arrow[r, squiggly] \arrow[d] & \mathcal{F}_M \arrow[d] \\ \mathbb{G}_m \circlearrowright \mathbb{A}^{n+1} \setminus 0 \arrow[r] & \mathbb{P}V \end{tikzcd}$$

(Note (1) ignores stuff supported at the irrelevant ideal).

Thm (r.g. proven version):

Let $\mathcal{C} \subset \mathrm{Sym}^\cdot V$-mod consist of those modules s.t. every elt. is annihilated by some power of the irrelevant ideal (V). Then

$\mathcal{F}_- : \mathrm{Sym}^\cdot V\text{-mod}/\mathcal{C} \to \mathrm{QCoh}(\mathbb{P}^n)$ is an equivalence,

localize at morphisms f w/ $\ker f, \mathrm{coker} f \in \mathcal{C}$

$\hat{\mathcal{F}}_- :$ f.g. $\mathrm{Sym}^\cdot V$-mod $\to \mathrm{Coh}(\mathbb{P}^n)$ restricts to an equivalence as well.

<u>Rem</u> Analogue for any graded (multi-graded) vty.

Reminder (Derived Category):

Start w/ an Abelian category A (R-mod, $\mathcal{O}_X$-mod, coherent sheaves on X, etc.)

$Ch^{\#}(A)$ is also an Abelian category (objects: chain complexes, morphisms: chain maps)

$\#$ is some decoration (bdd above or below, etc.)

$K^{\#}(A)$ - same objects, morphisms are chain maps / htpies
(triangulated category now!)

$D^{\#}(A)$ - localize $K^{\#}(A)$ or $Ch^{\#}(A)$ at quasi-isomorphisms.
hard! (explain why)

We are interested in $D^b_{Coh}(\mathbb{P}^n)$ — e.g. bdd cplexes of $\mathcal{O}_X$-modules w/ coherent cohomology.

Want to describe it as $K^{\#}(A)$ for some A — more computable

Will actually do this for two different A — $\mathrm{Sym}^\bullet V$ and $\Lambda^\bullet V$-mod.

Cohomological preliminaries:

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) = H^0(\mathbb{P}^n, \mathcal{O}(j-i)) = \mathrm{Sym}^{j-i} V$$

$$\mathrm{Ext}^\ell(\mathcal{O}(i), \mathcal{O}(j)) = H^\ell(\mathbb{P}^n, \mathcal{O}(j-i)) = 0, \quad -n \le i,j \le 0$$

(composition given by multiplication in $\mathrm{Sym}^\bullet V$)

Reminder (Euler exact sequence + relatives)

Want SES: $0 \to \Omega^i(i) \to \Lambda^i V \otimes \mathcal{O}_{\mathbb{P}^n} \to \Omega^{i-1}(i) \to 0$

$i=1$ - want $0 \to \Omega^1(1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$

Consider:

$$\mathbb{A}^{n+1}\setminus 0 \xrightarrow{\pi} \mathbb{P}V \rightsquigarrow$$
$$\downarrow$$
$$pt$$

$$0 \to \pi^*\Omega_{\mathbb{P}V} \to \underset{\substack{\cong \\ V^* \otimes \mathcal{O}_{\mathbb{A}^{n+1}\setminus 0}}}{\Omega_{\mathbb{A}^{n+1}\setminus 0}} \to \Omega_{\mathbb{A}^{n+1}\setminus 0/\mathbb{P}V} \to 0$$

Under the natural correspondence between $\mathbb{G}_m$-sheaves on $\mathbb{A}^{n+1}\setminus 0$ and sheaves on $\mathbb{P}V$, gives $i=1$ case.

General case:

Have $0 \to \Omega^1(1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$. Applying $\Lambda^i(-)$

get $0 \to \Omega^i(i) \to \Lambda^i(V) \otimes \mathcal{O}_{\mathbb{P}^n} \to \Omega^{i-1}(i) \to 0$

(b/c get filtration w/ associated graded

$\bigoplus_{p+q=i} \Lambda^p(\Omega^1(1)) \otimes \Lambda^q(\mathcal{O}_{\mathbb{P}^n}(1))$ — nonzero only for

$p=i,\ q=0 \rightsquigarrow \Omega^i(i)$

$p=i-1,\ q=1 \rightsquigarrow \Omega^{i-1}(i)$)

<u>Cor</u> $\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) = \Lambda^{i-j}(V^*)$

$\ell \geq 1$ $\mathrm{Ext}^\ell(\Omega^i(i), \Omega^j(j)) = 0$

Pf. Induction. $(j=0, i=0)$ ✓, (also: $\mathrm{Hom}(\Omega^i(i), \Omega^j(j-1)) = 0$, Ext, etc. etc.)

~~etc~~ General i:

$$0 \to \mathrm{Hom}(\Omega^{i-1}(i), \mathcal{O}) \to \underbrace{\mathrm{Hom}(\Lambda^i(V)\otimes\mathcal{O}, \mathcal{O})}_{\Lambda^i(V^*)} \to \mathrm{Hom}(\Omega^i(i), \mathcal{O}) \to \mathrm{Ext}^1(\Omega^{i-1}(i), \mathcal{O})$$

. . . $\square$

Now can state the theorem. If $A^\cdot$ is a graded algebra let

$A^\cdot[i]$ be the graded free $A^\cdot$-module w/ a generator in degree i.

Let $A^{[0,n]}$-mod be the category of finite direct sums of $A^\cdot[i]$ for $0 \leq i \leq n$.

Let $K^b(A^{[0,n]}\text{-mod})$ be the htpy category of bdd complexes of objects in $A^{[0,n]}$-mod.

Let $K_\Lambda = K^b(\Lambda^\cdot(V^*)^{[0,n]}\text{-mod})$

$K_S = K^b(\mathrm{Sym}^\cdot V^{[0,n]}\text{-mod})$

Thm (Beilinson)

Let $F_\Lambda : K_\Lambda \to D^b_{coh}(\mathbb{P}^n)$ send $\Lambda^\cdot V[i] \rightsquigarrow \Omega^i(i)$

$F_S : K_S \to D^b_{coh}(\mathbb{P}^n)$ send $\mathrm{Sym}^\cdot V[i] \rightsquigarrow \mathcal{O}(-i)$

and extend in the obvious way. Then

F_Λ, F_S are equivalences of categories.

Lemma Let C, D be triangulated categories and $F: C \to D$ an exact functor. Suppose $\{X_i\}$ generate C, $\{F(X_i)\}$ generates D, and $F: \mathrm{Hom}(X_i, X_j) \xrightarrow{\sim} \mathrm{Hom}(F(X_i), F(X_j))$ is an iso for all i, j. Then F is an equivalence.

So, want to show
$$\mathcal{O}(-i) \quad 0 \le i \le n$$
$$\Omega^i(i) \quad 0 \le i \le n$$
generate $D^b_{Coh}(\mathbb{P}^n)$.

Reminder: Koszul Resolution.

Local situation:

$I \subset R$, $(x_1, \dots, x_n) = I$ a regular sequence of generators (i.e. x_i is not a zero divisor in $R/(x_1 \dots x_{i-1})$).

Want to resolve R/I. Induction on generators:

$K^\bullet_{x_1} := 0 \to R \xrightarrow{x_1} R \to 0$ is a resolution of R/x_1

$$K^\bullet_{x_1} \otimes K^\bullet_{x_2} = \mathrm{Tot}\begin{pmatrix} \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & R \arrow[r, "\cdot x_1"] \arrow[d, "x_2"] & R \arrow[r] \arrow[d, "\cdot x_2"] & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot x_1"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd} \end{pmatrix} : 0 \to R \xrightarrow{\binom{x_1}{x_2}} R \oplus R \xrightarrow{(x_2\ x_1)} R \to 0$$

is a resolution of $R/(x_1, x_2)$

etc. (Rem. Exact iff $x_1 \dots x_n$ is regular, otherwise computes local cohomology).

Global Situation: $\begin{matrix} \mathcal{E} \\ \downarrow \\ X \end{matrix}$ is a vector bdle, $f \in \Gamma(X, \mathcal{E})$, want a resolution of $\mathcal{O}_{V(f)}$ by vector bdles:

$$\cdots \to \Lambda^3 \mathcal{E}^* \to \Lambda^2 \mathcal{E}^* \xrightarrow{\varphi} \underbrace{\mathcal{E}^* \to \mathcal{O}_X}_{\text{dual to}} \to \mathcal{O}_{V(f)} \to 0$$

(In the local situation $\mathcal{E} = R^n$, $f = (x_1 \dots x_n)$).

I claim the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$ is cut out by a natural section to $p_1^*\mathcal{O}(1) \otimes p_2^* T_{\mathbb{P}^n}(-1)$.

(In coordinates, this section is
$$\sum_i x_i \frac{\partial}{\partial y_i} \;)$$

Indeed, pulling back the Euler exact sequence in two different ways, we have:

$$0 \to p_1^*\Omega^1(1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to p_1^*\mathcal{O}_{\mathbb{P}^n}(1) \to 0$$
$$0 \to p_2^*\Omega^1(1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to p_2^*\mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

We may splice these sequences together to get a sequence,

$$0 \to p_2^*\Omega^1(1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \xrightarrow{s} p_1^*\mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

exact

exact only on $\Delta_{\mathbb{P}^n \times \mathbb{P}^n}$.

exact

This composite map is this canonical section $s \in \mathrm{Hom}(p_2^*\Omega^1(1), p_1^*\mathcal{O}_{\mathbb{P}^n}(1))$
$$\parallel$$
$$\Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*\mathcal{O}(1) \otimes p_2^* T_{\mathbb{P}^n}(-1))$$

The Koszul resolution now gives:

$$K_\Delta^\bullet : \cdots \to p_1^*\mathcal{O}(-3) \otimes p_2^*\Omega^3(3) \to p_1^*\mathcal{O}(-2) \otimes p_2^*\Omega^2(2) \to p_1^*\mathcal{O}(-1) \otimes p_2^*\Omega^1(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Will use this to construct a complex whose objects are direct sums of $\mathcal{O}(-i)$ $0 \le i \le n$, quasi-iso to any $X^\bullet$ or alternatively a cpx whose objects are direct sums of $\Omega^i(i)$, quasi-iso to any $X^\bullet$.

I'll do the latter — the former will be identical.

Consider X in $D^b(\mathbb{P}^n)$. Then

$K_\Delta^\bullet \overset{L}{\otimes} Lp_2^*(X)$ ($\simeq K_\Delta^\bullet \overset{L}{\otimes} p_2^*(X)$) by flatness of p_2, is a resolution of $p_2^* X$ and belongs to the full subcategory generated by stuff of the form $p_1^*(\Omega^i(i)) \otimes p_2^*(Y)$.

Then by the projection formula,

$$X = Rp_{1*}(K_\Delta^\bullet \overset{L}{\otimes} Lp_2^* X) \simeq Rp_{1*}(\mathcal{O}_\Delta \overset{L}{\otimes} Lp_2^* X) \quad \text{using } p_1^*(p_2^* X \otimes \mathcal{O}_\Delta)$$

But applying the projection formula termwise, this is a complex made up of objects of the form

$$\Omega^i(i) \otimes R\Gamma(Y), \text{ namely direct sums of } \Omega^i(i). \ \square$$

Let's make this more explicit, when X is a vector bdle.

unnecessary b/c X is flat w/ its sheaf then.

$$(K_\Delta \otimes_L Lp_1^* X) = \cdots \to p_2^*(\Omega^2(2)) \otimes p_1^*(X(-2)) \to p_2^*(\Omega^1(1)) \otimes p_1^*(X(-1)) \to p_1^*(X) \to p_1^*(X) \otimes \mathcal{O}_\Delta \to$$

unnecessary as p_1^* is flat

$$\cdots \to \quad \longrightarrow \mathcal{I}_2^\bullet \longrightarrow \mathcal{I}_1^\bullet \to \mathcal{I}_0^\bullet$$

$$\Big\downarrow Rp_{2*}$$

$$\mathrm{Tot}\left(\cdots \to Rp_{2*}\mathcal{I}_3^\bullet \to Rp_{2*}\mathcal{I}_2^\bullet \to Rp_{2*}\mathcal{I}_1^\bullet \to Rp_{2*}\mathcal{I}_0^\bullet \right)$$

E_1 page: $\to H^0(X(-2)) \otimes \Omega^2(2) \to H^0(X(-1)) \otimes \Omega^1(1) \to H^0(X) \otimes \mathcal{O} \to 0$

$$\cdots \to H^1(X(-2)) \otimes \Omega^2(2) \to H^1(X(-1)) \otimes \Omega^1(1) \to H^1(X) \otimes \mathcal{O} \to 0$$

$$\cdots \to H^2(X(-2)) \otimes \Omega^2(2) \to H^2(X(-1)) \otimes \Omega^1(1) \to H^2(X) \otimes \mathcal{O} \to 0$$

$$\vdots$$

<u>Remarks</u> (1) (Ancona, Ottaviani) Gluing together differentials from different levels of the SS, one gets a canonical ~~map~~ complex of direct sums of $\Omega^i(i)$ ~~exact away~~

$C_X^\bullet: 0 \to C^{-n} \to \cdots \to C^0 \to C^1 \to \cdots \to C^n \to 0$ from deg 0, w/ $H(C^\bullet) = \hat{X}$

(1') This is called a "monad" (remark on etymology).

(2) The composite equivalence $K_\Lambda \simeq D^b_{coh}(\mathbb{P}^n) \simeq K_S$ is not obvious!
This is the beginning of the BGG correspondence.
Useful for computer algebra (b/c $\Lambda^\bullet V$ is finite length!)
(See Eisenbud-Schreyer)

(3) Kapranov and others (e.g. Daniel Loughran at Berkeley) have generalized some of this to e.g. Grassmannians and other varieties.

(4) Useful for studying moduli of vector bdles (Horrocks) by parametrizing resolutions or monads (e.g. Matrix factorizations) (inspiration for previous talk).