DUAL CANTOR-SCHROEDER-BERNSTEIN FOR RINGS

DANIEL LITT

Sam Lichtenstein poses the following question:

> Do there exist finite-type k-algebras A and B not isomorphic to one another, and surjections
> $$f : A \to B, g : B \to A?$$

We answer this question in the negative; indeed, we show the slightly more general claim

Proposition 1. *Let A, B be Noetherian rings, with surjections $f : A \to B, g : B \to A$. Then f, g are isomorphisms.*

Proof. First, consider the composition $A \xrightarrow{f} B \xrightarrow{g} A$; note that this map is surjective, and call its kernel I. We assume for the sake of contradiction that I is non-zero.

We have an isomorphism $s : A/I \xrightarrow{\sim} A$. Let $\pi : A \to A/I$ be the projection. Then $I_2 := \pi^{-1}(s^{-1}(I)) \subset A$ is an ideal of A strictly containing I. Furthermore, we have $(A/I)/(s^{-1}(I)) \simeq A/I \simeq A$, and thus $A/I_2 \simeq (A/I)/(s^{-1}(I)) \simeq A/I \simeq A$. Let $I_n = \pi^{-1}(s^{-1}(I_{n-1}))$. Then $I \subset I_2 \subset I_3 \subset ...$ is an ascending chain of ideals in A which does not stabilize, contradicting Noetherian-ness.

So I is zero, and thus $g \circ f$ is an isomorphism. Thus, f is injective, and is an isomorphism; the same argument shows that g is an isomorphism. $\square$

We also note that following easy corollary, which expresses the content of the proof.

Corollary 1. *A Noetherian ring is not isomorphic to any proper quotient of itself.*

Geometrically, the idea of this argument is that if $\mathrm{Spec}(A)$ is isomorphic to a proper subscheme of itself, say X, then X is isomorphic to a proper subscheme of itself X', and so on. But that's impossible by Noetherian-ness.