FULTON'S TRACE FORMULA

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1. Introduction

Let X be a projective variety over a finite field $\mathbb{F}_q$ of characteristic p. Then the zeta function of $X/\mathbb{F}_p$ is defined as

$$Z(X,t) = \exp\left(\sum_{i=1}^{\infty} \frac{\#|X(\mathbb{F}_{q^k})|}{k} t^k\right) = \prod_{x \in X_{cl}} (1 - t^{\deg(x)})^{-1}$$

where the product on the right is taken over the closed points of x, and the equality follows by Galois theory. $Z(X,t)$ is the generating function for the number of effective zero-cycles on X defined over $\mathbb{F}_q$; its logarithmic derivative is the generating function for the number of $\mathbb{F}_{q^k}$-rational points of X.

Grothendieck defines ℓ-adic cohomology groups $H^i(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ satisfying the following beautiful identity:

Theorem 1 (Grothendieck)**.**

$$\#|X(\mathbb{F}_{q^k})| = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Tr}(\mathrm{Frob}_q^k \mid H^i(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)) \qquad (1)$$

and thus

$$Z(X,t) = \prod_{i=0}^{2\dim(X)} \det(1 - t\,\mathrm{Frob}_q \mid H^i(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}. \qquad (2)$$

The first of these formulas is analogous to the Lefschetz fixed-point formula; the latter is a formal consequence. The development of ℓ-adic cohomology is difficult (albeit necessary to develop a cohomology theory with characteristic-zero coefficients for varieties in characteristic p); the goal of this talk is to describe a similar formula in *coherent* cohomology.

This formula is due in various forms to Fulton, Katz, and Deligne.

Theorem 2. *Let X be a projective variety over a finite field $\mathbb{F}_q$, of characteristic p. Then*

$$\#|X(\mathbb{F}_{q^k})| \equiv \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{Tr}(\mathrm{Frob}_q^k \mid H^i(X, \mathcal{O}_X)) \bmod p.$$

In SGA $4\frac{1}{2}$ and SGA 7, Deligne and Katz push this technology further to prove a determinental formula analogous to Theorem 1, (2) for $Z(X,t) \bmod p$. My presentation will follow that of Mustata and Fulton.

2. Preliminaries on Frobenius

Let X be a scheme over $\mathbb{F}_q$. If $X = \operatorname{Spec}(A)$ is affine, it admits a natural endomorphism, denoted Frob_q, via

$$\mathrm{Frob}_q : A \to A$$
$$x \mapsto x^q.$$

This induces a map

$$\mathrm{Frob}_q : \operatorname{Spec}(A) \to \operatorname{Spec}(A),$$

respecting the structure map $A \to \operatorname{Spec}(\mathbb{F}_q)$, and so is an endomorphism; furthermore, if $X' = \operatorname{Spec}(A) \cup \operatorname{Spec}(B)$, and given any affine $\operatorname{Spec}(C) \subset \operatorname{Spec}(A) \cap \operatorname{Spec}(B)$, then the natural restriction maps $A \to C, B \to C$, commute with Frob_q and so Frob_q extends to an endomorphism of any scheme over $\mathbb{F}_q$. Furthermore,

Frob_q commutes with morphisms between schemes; e.g. if $f : X \to Y$ is a morphism over $\mathrm{Spec}(\mathbb{F}_q)$, then $\mathrm{Frob}_q \circ f = f \circ \mathrm{Frob}_q$. Put succinctly, Frob_q is an endomorphism of the identity functor on $\mathrm{Sch}/\mathbb{F}_q$.

Viewed strictly as a map of topological spaces,

$$\mathrm{Frob}_q : X \to X$$

is simply the identity, and so in particular, if M is a coherent sheaf on X, $\mathrm{Frob}_{q*}(M)$ is naturally isomorphic to M as a sheaf of Abelian groups. However, as an $\mathcal{O}_X$-module, it is rather different; if the action $\mathcal{O}_X(U) \otimes M(U) \to M(U)$ is given by

$$x \otimes m \mapsto xm,$$

the action $\mathcal{O}_X(U) \otimes \mathrm{Frob}_{q*}(M)(U) \to \mathrm{Frob}_{q*}(M)(U)$ is given by

$$x \otimes m \mapsto x^q m.$$

Note however that this does not affect cohomology (this is a subtle point, so make sure you buy it before you move on), and so

$$H^i(X, M) \simeq H^i(X, \mathrm{Frob}_{q*}(M))$$

naturally.

Now there is a natural map $\mathcal{O}_X \to \mathrm{Frob}_{q*}(\mathcal{O}_X)$ which is part of the data of the map $\mathrm{Frob}_q : X \to X$. This induces a map on cohomology

$$H^i(\mathrm{Frob}_q) : H^i(X, \mathcal{O}_X) \to H^i(X, \mathrm{Frob}_{q*}(\mathcal{O}_X)) \simeq H^i(X, \mathcal{O}_X);$$

it is this map whose traces we will study.

Let us generalize our set-up slightly. A coherent Frobenius module, or F-module for short, is a coherent sheaf $\mathcal{M}$ on X equipped a map of $\mathcal{O}_X$-modules $F : \mathcal{M} \to \mathrm{Frob}_{q*}(\mathcal{M})$; namely for $x \in \mathcal{O}_X(U)$, $m \in \mathcal{M}(U)$, we will have $F(xm) = x^q F(m)$. Obviously, $\mathcal{O}_X$ with the map induced by Frobenius above is an F-module. Furthermore, we have as above a map

$$H^i(F) : H^i(X, \mathcal{M}) \to H^i(X, \mathcal{M}).$$

While the identification $\mathcal{M} \simeq \mathrm{Frob}_{q*}(\mathcal{M})$ is not a map of $\mathcal{O}_X$-modules, it is a map of $\mathbb{F}_q$ modules. Thus if $x : \mathrm{Spec}(\mathbb{F}_q) \to X$ is an $\mathbb{F}_q$-point, there is a natural map of stalks

$$x^* F : x^*\mathcal{M} \to x^*\mathrm{Frob}_{q*}\mathcal{M} \simeq x^*\mathcal{M},$$

which we will henceforth denote

$$F(x) : \mathcal{M}(x) \to \mathcal{M}(x).$$

A morphism of F-modules is a map $g : \mathcal{M} \to \mathcal{N}$ so that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "g"] \arrow[d, "F_{\mathcal{M}}"] & \mathcal{N} \arrow[d, "\mathcal{F}_{\mathcal{N}}"] \\ \mathrm{Frob}_{q*}\mathcal{M} \arrow[r, "\mathrm{Frob}_{q*} g"] & \mathrm{Frob}_{q*}\mathcal{N} \end{tikzcd}$$

commutes. The category of F-modules is an Abelian category.

We're now ready to state a generalization of Theorem 2.

Theorem 3. *Let $(\mathcal{M}, F)$ be an F-module on X. Then*

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(F(x)) = \sum_{i=0}^{\dim(X)} (-1)^i \,\mathrm{Tr}(F \mid H^i(X, \mathcal{M})).$$

Let's deduce our original theorem from this one. For $k = 1$, we wish to show

$$\#|X(\mathbb{F}_q)| \equiv \sum_{i=0}^{\dim(X)} (-1)^i \,\mathrm{Tr}(\mathrm{Frob}_q \mid H^i(X, \mathcal{O}_X)) \bmod p.$$

But for $x \in X(\mathbb{F}_q)$, $\mathcal{O}_X(x) = \mathbb{F}_q$ and $\mathrm{Frob}_q(x) = \mathrm{Id}_{\mathbb{F}_q}$, so the sum on the left is $\#|X(\mathbb{F}_q)| \bmod p$, as desired. For $k > 1$, the result follows by base-changing to $\mathbb{F}_{q^k}$ and applying the result with $k = 1$.

Note that the quantities in Theorem 3 depend on much less than the isomorphism class of $(\mathcal{M}, F)$, as they are additive in F and along short exact sequences of F-modules. Thus, we make the following definition.

Definition 1 (Grothendieck Group of F-modules)**.** *Let X be a projective scheme over $\mathbb{F}_q$. Then $K^F(X)$ is the free abelian group on isomorphism classes of F-modules, subject to the relations:*

- $(\mathcal{M}, F') + (\mathcal{M}, F'') = (\mathcal{M}, F' + F'')$
- *If*
$$0 \to (\mathcal{M}', F') \to (\mathcal{M}, F) \to (\mathcal{M}'', F'') \to 0$$
is a short exact sequence, then $(\mathcal{M}, F) = (\mathcal{M}', F') + (\mathcal{M}'', F'')$.

By the remarks above, both quantities in Theorem 3 depend only on the class of $(\mathcal{M}, F)$ in $K^F(X)$—indeed, they are homomorphims $K^F(X) \to \mathbb{F}_q$—so we may as well study $K^F(X)$ as an approach to proving our theorem. Note that $\mathbb{F}_q$ acts on $K^F(X)$ by multiplication, so $K^F(X)$ is an $\mathbb{F}_q$-vector space.

Proposition 1. *The following are properties of K^F.*

(1) *Let (M, F_M) be an F-module on X with F_M nilpotent. Then the class of (M, F_M) is zero in $K^F(X)$.*
(2) *Let (M, F) satisfy $M = M_1 \oplus \cdots \oplus M_n$; let F_{ij} be the composition $F_{ij} : M_i \to M \xrightarrow{F} M \to M_j$. Then the class of (M, F) is equal to that of $\sum_i (M_i, F_{ii})$.*
(3) *$K^F(\mathrm{Spec}(\mathbb{F}_q)) = \mathbb{F}_q$, via the map*
$$(\mathcal{M}, F) \mapsto \mathrm{Tr}(F(x))$$
with x the unique point of $\mathrm{Spec}(\mathbb{F}_q)$.
(4) *If X is the disjoint union of connected components $X_1, \cdots, X_r$, then*
$$K^F(X) = \bigoplus K^F(X_i).$$
(5) *If $j : X \to Y$ is a closed embedding, j^* induces a map $K^F(Y) \to K^F(X)$, functorial in j. (K^F is also contravariantly functorial for flat morphisms, but we won't use this.)*
(6) *If $f : X \to Y$ is a* proper *map of projective schemes over $\mathbb{F}_q$, $\sum (-1)^i R^i f_*$ induces a map $K^F(X) \to K^F(Y)$, functorial in f.*
(7) *If $j : X \to Y$ is a closed embedding, then $j^* \circ j_* : K^F(X) \to K^F(X)$ is the identity.*

Proof. (1) Note that if $F_M = 0$, then $(M, F_M) = (M, F_M) + (M, F_M)$ and thus equals zero. Now suppose $F_M^n = 0$. Then there is a short exact sequence
$$0 \to (\ker(F_M), 0) \to (M, F_M) \to (\mathrm{im}(F_M), F_M|_{\mathrm{im}(F_M)}) \to 0$$
so $(M, F_M) = (\mathrm{im}(F_M), F_M|_{\mathrm{im}(F_M)})$. But $F_M|_{\mathrm{im}(F_M)}^{n-1} = 0$, so the result follows by induction.

(2) Let $\phi_{ij} : M \to M$ be the map $M \to M_i \xrightarrow{F_{ij}} M_j \to M$; then $(M, F) = \sum_{ij} (M, \phi_{ij})$. If $i \neq j$, then $\phi_{ij}^2 = 0$ and so $(M, \phi_{ij}) = 0$ for $i \neq j$ by (1). On the other hand,
$$(M, \phi_{ii}) = (M_i, F_{ii}) + \sum_{j \neq i} (M_j, 0) = (M_i, F_{ii}),$$
completing the proof.

(3) An F-module on $\mathrm{Spec}(\mathbb{F}_q)$ is a pair (V, F) with V a finite-dimensional $\mathbb{F}_q$-vector space and $F : V \to V$ an endomorphism. Writing V as a direct sum of one-dimensional vector space gives the result immediately from (2) above.

(4) Trivial.

(5) Let I be the ideal sheaf of X; then $j^*(M) = M/IM$; $F_M(IM) \subset I^q F_M(M) \subset IM$, so F_M induces a map $\overline{F_M} : j^*(M) \to j^*(M)$; we must check that this map respects the relations on $K^F(X)$. If $F_M = F'_M + F''_M$, then $\overline{F_M} = \overline{F'_M} + \overline{F''_M}$, by definition.

Let
$$0 \to (M', F'_M) \to (M, F_M) \to (M'', F''_M) \to 0$$
be a short exact sequence of F-modules on Y. Then there are short exact sequences
$$0 \to M'/(M' \cap IM) \to M/IM \to M''/IM'' \to 0$$

and

$$0 \to (M' \cap IM)/IM' \to M'/IM' \to M'/(M' \cap IM) \to 0.$$

It suffices to show that $M'/IM', M'/(M' \cap IM)$ are equal in $K^F(X)$, and thus it suffices to show that $(M' \cap IM)/IM'$ is zero in $K^F(X)$. Indeed, Artin-Rees shows that the Frobenius action on $(M' \cap IM)/IM'$ is nilpotent, so the result follows by (1).

(6) That the map is well-defined follows from proper base change and the usual properties of derived functors; functoriality follows from the Leray spectral sequence.

(7) This follows from the definition, as $R^i j_* = 0$ for $i > 0$.

□

3. The Localization Theorem

Let's refine our theorem once more.

Theorem 4 (Localization). *Let X be a projective scheme over $\mathbb{F}_q$, and let $j : X(\mathbb{F}_q) \to X$ be inclusion of the closed subscheme of X consisting of those closed points with residue field $\mathbb{F}_q$. Then $j_* : K^F(X(\mathbb{F}_q)) \to K^F(X)$ is an isomorphism, with inverse j^*.*

Let's be a bit more explicit. By Proposition 1, (3) and (4), $K^F(X(\mathbb{F}_q)) = \mathbb{F}_q^{\#|X(\mathbb{F}_q)|}$. As j_* is exact, the map $j_* : K^F(X(\mathbb{F}_q)) \to K^F(X)$ is the usual pushforward. Also, by Proposition 1, (3), the map $j^* : K^F(X) \to K^F(X(\mathbb{F}_q))$ is given by

$$(M, F) \mapsto \sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(F(x)) \cdot \langle x \rangle,$$

where $\langle x \rangle$ is the class $((\mathbb{F}_q)_x, \mathrm{id})$ i.e. a skyscaper sheaf at x with stalk $\mathbb{F}_q$, with the identity map (this is the structure sheaf of x as a closed subscheme of $X(\mathbb{F}_q)$). Note that $j^* j_* = \mathrm{id}$ by Proposition 1, (7).

To see that Theorem 3 follows from the localization theorem, consider the triangle

$$\begin{tikzcd}
K^F(X(\mathbb{F}_q)) \arrow[rr, bend left, "j_*"] \arrow[dr, "s_*"'] & & K^F(X) \arrow[ll, bend left, "j^*"] \arrow[dl, "\pi_*"] \\
& K^F(\mathrm{Spec}(\mathbb{F}_q)) &
\end{tikzcd}$$

where s, π are the structure maps of $X(\mathbb{F}_q), X$ respectively. A priori the "clockwise" triangle commutes (e.g. the path involving j_*), whereas the "counter-clockwise" triangle (involving j^*) commutes given the theorem.

Let (M, F) be an F-module on X. Then

$$\pi_*(M, F) = \sum_{i=0}^{\dim(X)} (-1)^i \,\mathrm{Tr}(F \mid H^i(X, M))$$

after identifying $K^F(\mathrm{Spec}(\mathbb{F}_q))$ with $\mathbb{F}_q$. On the other hand, $\pi_* = \pi_* j_* j^* = s_* j^*$ using Theorem 4. Explicitly, s_* is given by

$$s_* \left(\sum_{x \in X(\mathbb{F}_q)} m_x \cdot \langle x \rangle \right) = \sum_{x \in X(\mathbb{F}_q)} m_x.$$

Combining this with our description of j^* above gives

$$\pi_*(M, F) = s_* j^*(M, F) = \sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(F(x))$$

and equating these two quantities gives the desired result.

Finally, we turn to the proof of Theorem 4. The idea will be to show the result for projective space, and then to deduce it for all projective varieties.

Lemma 1. *Let $j : X \to Y$ be a closed embedding of varieties over $\mathbb{F}_q$. If $i' : Y(\mathbb{F}_q) \to Y$ satisfies $i'_* : K^F(Y(\mathbb{F}_q)) \to K^F(Y)$ is an isomorphism, with inverse i'^*, then $i_* : K^F(X(\mathbb{F}_q)) \to K^F(X)$ is an isomorphism with inverse i^*.*

Proof. As $i^*i_* = \mathrm{id}$ in general, it suffices to show that $i_*i^* = \mathrm{id}$. Consider the diagram

$$\begin{tikzcd} X(\mathbb{F}_q) \arrow[r, "j'"] \arrow[d, "i"] & Y(\mathbb{F}_q) \arrow[d, "i'"] \\ X \arrow[r, "j"] & Y \end{tikzcd}$$

Note that ${i'}^* \circ j_* = {j'}_* \circ i^*$ (using that all the arrows are closed embeddings). Thus

$$j_*i_*i^* = i'_*j'_*i^* = i'_*{i'}^*j_* = j_*,$$

and thus composing on the left with j^* gives

$$i_*i^* = \mathrm{id}$$

as desired. $\square$

Now it suffices to prove Theorem 4 for $X = \mathbb{P}^n_{\mathbb{F}_q}$. To do this, we essentially compute $K^F(\mathbb{P}^n_{\mathbb{F}_q})$.

4. The Localization Theorem for $\mathbb{P}^n$

First, let's analyze how Frob_q^* acts on line bundles. We claim that if L is a line bundle on X, ${\mathrm{Frob}_q}^*(L) \simeq L^q$. To see this, let $\mathrm{Spec}(A), \mathrm{Spec}(B)$ be open affines of X on which L is trivial, and let $\mathrm{Spec}(C) \subset \mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ be an affine contained in their intersection. Then the two different trivializations of L over $\mathrm{Spec}(A), \mathrm{Spec}(B)$ give an automorphism of C as a C-module, i.e. multiplication by a unit $c \in C^\times$. Applying Frob_q^*, c acts via c^q, giving the desired claim.

We now recall Serre's correspondence between coherent sheaves over $\mathbb{P}^n$ and finitely generated $\mathbb{Z}$-graded $k[x_0, ..., x_n]$-modules. Let $S = \mathbb{F}_q[x_0, ..., x_n]$, and let M be a finitely generated graded S-module. Then we obtain a coherent sheaf $\tilde{M}$ over $\mathbb{P}^n$ by setting $\tilde{M}(U_f) = (M_f)_0$ for f a homogenous polynomial and U_f the complement of its vanishing set. To go the other way, we send a coherent sheaf to the graded S-module

$$\bigoplus_i \Gamma(\mathbb{P}^n, M(i)).$$

Now consider an F-module (M, F) on $\mathbb{P}^n_{\mathbb{F}_q}$. M is by definition equipped with a map

$$F : M \to {\mathrm{Frob}_q}_*(M)$$

which, by tensoring with $\mathcal{O}(n)$ induces a map

$$F : M(n) \to {\mathrm{Frob}_q}_*(M) \otimes \mathcal{O}(n) \simeq {\mathrm{Frob}_q}_*(M \otimes {\mathrm{Frob}_q}^*(\mathcal{O}(n))) \simeq {\mathrm{Frob}_q}_*(M(qn))$$

using the projection formula and the fact that ${\mathrm{Frob}_q}^*(L) \simeq L^q$ for L a line bundle. It is thus straightforward to see that translating the language of F-modules into the land of graded S-modules, an F-module is a graded S-module M with a map $F : M \to M$ satisfying $F(M_r) \subset M_{qr}$ and $F(ax) = a^qF(x)$ for $a \in S, x \in M$.

In particular, an F-module structure on $\mathcal{O}(-r) = \widetilde{S(-r)}$ is given by a map $F : S(-r) \to S(-r)$, and is determined by $F(1) \in S(-r)_{rq} = S_{r(q-1)}$. In particular, if $r < 0$, the only such structure is the zero structure.

Now let (M, F) be any F-module on $\mathbb{P}^n_{\mathbb{F}_q}$. By the Hilbert Syzygy theorem we may find a resolution

$$0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to M \to 0$$

where each

$$E_i \simeq \bigoplus_j \mathcal{O}(-n_{i,j})$$

for some $n_{i,j}$. We may choose a lift of F to E_0, and continuing inductively we may equip the E_j with an F-module structure so that the sequence above is a resolution of M by F-modules. Thus $M = \sum_{i=0}^n (-1)^i E_i$, and so $K^F(\mathbb{P}^n_{\mathbb{F}_q})$ is generated by classes of the form $(\mathcal{O}(-n), f)$ with $n \geq 0, f$ a homogeneous polynomial of degree $n(q-1)$. Indeed, we may take f to be a monomial, by additivity.

We refine this description of $K^F(\mathbb{P}^n_{\mathbb{F}_q})$.

Lemma 2. *$K^F(\mathbb{P}^n_{\mathbb{F}_q})$ is generated by classes of the form $(\mathcal{O}(-r), x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n})$, with $\sum a_i = r(q-1), 0 \leq a_i \leq q-1$, with at least one a_i strictly less than $q-1$.*

Proof. We first show that if some $a_i > q-1$, we may decrease it by $q-1$. Namely, let there is an exact sequence of F-modules

$$0 \to (\mathcal{O}(-i), x_i^q w) \xrightarrow{\cdot x_i} (\mathcal{O}(-i+1), x_i w) \to (\mathcal{O}_{V(x_i)}(-i+1), x_i w) \to 0$$

where $\mathcal{O}_{V(x_i)}$ is the structure sheaf of the hyperplane cut out by x_i. But the Frobenius action on this last is 0, as x_i vanishes on $V(x_i)$, and so $(\mathcal{O}(-i), x_i^q w) = (\mathcal{O}(-i+1), x_i w)$ in $K^F(\mathbb{P}^n_{\mathbb{F}_q})$.

Now we must show that if each $a_i = q-1$, we can write the resulting F-module $(\mathcal{O}(-n-1), x_0^{q-1} x_1^{q-1} \cdots x_n^{q-1})$ in terms of the other generators. We do this using the Koszul complex for the global sections $x_0, ..., x_n$ of $\mathcal{O}(1)$ We may write this as the tensor product of the chain complexes

$$\mathcal{C}^i_\bullet : 0 \to \mathcal{O}(-1) \xrightarrow{\cdot x_i} \mathcal{O} \to 0$$

over all i, where the Frobenius action on the $\mathcal{O}(-1)$ appearing in $\mathcal{C}^i_\bullet$ is given by x_i^{q-1}. Then writing this tensor product as

$$0 \to E_{n+1} \to E_n \to \cdots \to E_1 \to E_0 = \mathcal{O} \to 0$$

we see that

$$E_r = \bigoplus_{0 \le i_1 \le \cdots \le i_r \le n} \left(\bigotimes_{1 \le j \le r} (\mathcal{O}(-1), x_{i_j}^{q-1}) \right) = \bigoplus_{0 \le i_1 \le \cdots \le i_r \le n} (\mathcal{O}(-r), x_{i_1}^{q-1} \cdots x_{i_r}^{q-1}).$$

This sequence is exact, and $E_{n+1} = (\mathcal{O}(-n-1), x_0^{q-1} x_1^{q-1} \cdots x_n^{q-1})$, so this gives

$$\sum_i (-1)^i E_i = 0$$

as the desired relation in $K^F(\mathbb{P}^n_{\mathbb{F}_q})$. □

Corollary 1. *The dimension of $K^F(\mathbb{P}^n_{\mathbb{F}_q})$ as an $\mathbb{F}_q$-vector space at most*

$$1 + q + q^2 + \cdots + q^n$$

Proof. There are $q^n - 1$ monomials $x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ with $0 \le a_i \le q-1$, with at least one $a_i < q-1$. The lemma above gives a generating set of $K^F(\mathbb{P}^n_{\mathbb{F}_q})$ in bijection with those monomials satisfying the conditions above, as well as the condition

$$\sum a_i \equiv 0 \bmod q-1.$$

Let b_n be the number of monomials satisfying all these conditions, and let $\prod x_i^{a_i}$ be one such monomial. If $\sum_{0 \le i \le n-1} a_i \not\equiv 0 \bmod q-1$, or if all $a_i = q-1$ for $i < n$, then a_n is completely determined; otherwise a_n must equal 0 or $q-1$. Thus $b_n = 2b_{n-1} + (q^n - b_{n-1}) = q^n + b_{n-1}$. As $b_0 = 1$, this gives

$$b_n = 1 + q + q^2 + \cdots + q^n$$

as desired. □

Finally, we can complete the proof of the localization theorem.

Proof of Localization. We wish to show that $j_* : K^F(\mathbb{P}^n(\mathbb{F}_q)) \to K^F(\mathbb{P}^n)$ is an isomorphism, with inverse j^*. We already know that $j^* j_*$ is the identity, so j_* is injective. But the dimension of $K^F(\mathbb{P}^n(\mathbb{F}_q))$ as an $\mathbb{F}_q$-vector space is $\#|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \cdots + q^n$, so by the corollary above, j_* is surjective, and thus an isomorphism. Combined with Lemma 1 above, this gives the result for arbitrary projective schemes over $\mathbb{F}_q$. □