GEOMETRIZING COHOMOLOGY

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1. Introduction

[A quick note—I'd like to try an experiment and make this talk "interactive." At times I'll ask for input, but please feel free to interject any time I take a breath. Also, this will be a very soft talk.]

Let's start by interpreting $H^0(X, \mathbb{Z})$, where X is some nice topological space. Here are some ways of looking at this group.

- (Homotopy classes of) maps $X \to \mathbb{Z}$.
- Exactly the same thing: $\Gamma(X, \underline{\mathbb{Z}})$.
- Definition from singular cohomology...
- If X is a compact oriented n-manifold—a map from another compact oriented n-manifold $M \to X$ gives a cohomology class, via Poincar'e duality...
- [Any others?]

I'm listing these equivalent definitions because some of them are much more "geometric" than others. The goal of this talk will be to give geometric interpretations of higher cohomology classes. Let's try $H^1(X, \mathbb{Z})$:

- Homotopy classes of maps $X \to S^1$
- $\mathbb{Z}$-torsors over X
- Cech cohomology, singular cohomology, etc
- If M is a compact oriented manifold, a map in from a codimension 1 compact oriented manifold. (Up to some equivalence relation...)
- [Any others?]

Things start to get pretty interesting at H^2, especially if we add more structure. Here are some interpretations of $H^2(X, \mathbb{Z})$.

- Homotopy classes of maps $X \to \mathbb{CP}^\infty$
- Complex line bundles (either via the above or via the SES
$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$
or via Cech cohomology.
- Cech cohomology, singular cohomology, etc
- If X is a compact oriented manifold, a map in from a codimension 2 compact oriented manifolds gives an element of $H^2(X, \mathbb{Z})$...
- If X is a complex manifold and $V \subset X$ is a complex codimension 1 submanifold, one gets a holomorphic line bundle $\mathcal{L}$ on X, along with a canonical holomorphic section (up to scaling)—this gives a class in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$.
- Similarly, if $\gamma \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, γ can be viewed as living in $H^1(X, \Omega^1_X)$, whence it is an Ω^1_X-torsor. In the previous interpretation, it is precisely the Ω^1_X-torsor of holomorphic connections on X.

There's a standard way in which some of these pictures generalize. For example, one may view an element of $H^n(X, \mathbb{Z})$ as:

- Singular cohomology, Cech cohomology, etc.
- Maps $X \to K(\mathbb{Z}, n)$
- "Principal $K(\mathbb{Z}, n-1)$-bundles" (but there's some subtlety as to what this means)
- Similar interpretation with maps in from codimension n manifolds...

These interpretations have some disadvantages—here are two:

- In the case of a complex line bundle on a smooth manifold, we may find an explicit differential form representative of its Chern class (the curvature), after choosing a connection. These explicit representatives are hard to build in the topological setting.
- We may naturally associate to each divisor on a complex manifold a holomorphic line bundle and a meromorphic section (defined up to scaling). This is useful for several reasons—first, the natural equivalence relation on divisors is tractable, and second, the moduli of divisors is related to the moduli of line bundles.

Before moving on to imitate these constructions on more generality, let me sketch them for line bundles.

First, suppose $\mathcal{L}$ is a complex line bundle on a manifold M. Recall that a connection is a "way of differentiating sections to $\mathcal{L}$," namely a map $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1(X)$ which is compatible with the exterior derivative, in the sense that $\nabla(fs) = s \otimes df + f\nabla(s)$. [Explain why this gives a way of differentiating.] In a local trivialization U (say with trivializing—that is, non-vanishing—section s, we may (canonically) view $\nabla(s)/s$ as a $\mathbb{C}$-valued function 1-form on U. Then $d(\nabla(s)/s)$ is the curvature of $(\mathcal{L}, \nabla)$.

Another way of looking at this is as follows: choose a cover $\{U_i\}$ of X, and a trivialization s_i of $\mathcal{L}$ on each U_i. Choose 1-forms ω_i on U_i so that $\omega_i|_{U_{ij}} = df_{ij} + \omega_j|_{U_{ij}}$. Then we may set $\nabla(s_i) = \omega_i$. Here's one more perspective—if $\gamma : [0,1] \to X$ is a path with $\gamma(0) = x$, and y is a point in $\mathcal{L}_x$, we may lift γ to a canonical path γ' in $\mathcal{L}$, with $\gamma(0) = y$,, and $\nabla(\gamma')(D\gamma) = 0$. [Explain what this means.] Then the curvature α of $(\mathcal{L}, \nabla)$ may be interpreted as follows:

If $H : [0,1]^2 \to X$ is a (based) homotopy between paths γ and λ (between points x_1 and x_2), then the maps $\mathcal{L}_x \to \mathcal{L}_y$ induced by γ and λ differ by

$$e^{\int_{[0,1]^2} H^*(\alpha)}.$$

Here's one more point of view on line bundles with connection. Define

$$\mathbb{Z}(p)_D := 0 \to (2\pi i)^p\mathbb{Z} \to \mathcal{A}^0_X \to \mathcal{A}^1_X \to \cdots \to \mathcal{A}^{p-1}_X \to 0$$

in the smooth setting, and

$$\mathbb{Z}(p)_D := 0 \to (2\pi i)^p\mathbb{Z} \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X \to 0$$

in the holomorphic setting. Here $\mathcal{A}^i_X$ denotes the sheaf of smooth i-forms on a manifold X, and Ω^i_X denotes the sheaf of holomorphic i-forms on a complex manifold X. In the smooth setting, $H^2(X, \mathbb{Z}(2)_D)$ is the group of complex line bundles with connection; in the holomorphic setting, this is the group of holomorphic line bundles with holomorphic connection. [Explain Cech hypercohomology...]

I'd also like to review the construction of a line bundle from a divisor; just for simplicity, I'll stick with the case of a divisor on a Riemann surface. [Explain construction, Abel-Jacobi map].

Note that we may interpret the Abel-Jacobi map on a smooth proper complex curve X as a map

$$\coprod_n \mathrm{Sym}^n(X) \to H^2(X, \mathbb{Z}(2)_D),$$

and that both the left and the right hand side may be interpreted as algebraic varieties. This is the first example of what I mean by geometrizing cohomology. In fact, if X is an n-dimensional compact complex manifold, there is always a "cycle class map"

$$\{\text{codimension d cycles on } X\} \to H^{2d}(X, \mathbb{Z}(d)_D)$$

though the thing on the right need not be an algebraic variety. We'd like to interpret these maps as a way of passing between cycles and "some object like a line bundle with connection."

Let's start thinking about $H^3(X, \mathbb{Z})$ a little bit. Here are some things that this group can be interpreted as:

- Singular/Cech cohomology, maps to $K(\mathbb{Z}, 3)$, codimension 3 manifolds mapping in, etc.
- $PU(H)$-bundles on X, for H an infinite-dimensional Hilbert space [explain this—the content is that $PU(H)$ is a $K(\mathbb{Z}, 2)$]
- $\mathcal{O}^*_X$-banded gerbes

This last interpretation is the one I'd like to focus on for the rest of the talk.

2. What is a gerbe?

Let G be a group. The typical example of a G-banded gerbe is the stack of principal G-bundles. [Explain the general idea of a stack.] What are some of its properties:

- Any two principal G-bundles are locally isomorphic; isomorphisms between them form a G-torsor.
- About any point in X, there's an open set and a principal G-bundle on that set.
- Descent, etc. (it's a stack)

You should think of a G-banded gerbe as a stack which is locally equivalent to the stack of principal G-bundles.

So let's figure out (1) how to present a gerbe, and (2), how they are related to cohomology. Thinking in terms of sheaf cohomology, we may view a class in $H^3(X, \mathbb{Z})$ as coming from $H^2(X, \mathcal{O}_X^*)$, and thus as being represented by a Cech cocycle assigning non-vanishing functions to triple intersections in some cover $\{U_i\}$ of X, which satisfy some cocycle condition. Thinking carefully, these give an element of $H^1(U_i \cap U_j, \mathcal{O}_X^*)$ on each intersection; that is, a line bundle! So we can present a gerbe as being given by line bundles $\mathcal{L}_{ij}$ on U_{ij}, as well as the data of isomorphisms $\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \to \mathcal{L}_{ik}$, which satisfy a coherence condition on quadruple intersections. How does this relate to the point of view of a gerbe as a stack?

On an open set U, we consider the category of collections of line bundles $\mathcal{M}_i$ on $U \cap U_i$, as well as isomorphisms $M_i|_{U \cap U_{ij}} \otimes \mathcal{L}_{ij} \to M_j|_{U \cap U_{ij}}$ satisfying a coherence conditions, with isomorphisms given in the obvious way. This gives a gerbe in the stacky sense.

I'd like to explain how to go from a point in a complex surface (2 complex dimensions) or Riemannian 3-fold, to a gerbe.

2.1. The holomorphic setting.

The general plan in the holomorphic category is as follows. For concreteness, suppose X is a complex surface (four real dimension). Then we define $\mathcal{G}_X^2$ to be the 2-stack of "meromorphic $\mathcal{O}_X^*$-banded gerbes on X," namely gerbes defined away from the complement of some collection of points. Given a 0-cycle $D = \sum a_i[x_i]$, we wish to take the sub-2-stack $\mathcal{G}(D)$ of $\mathcal{G}_X^2$ consisting of meromorphic gerbes which have order at most a_i at x_i (where the order of a meromorphic gerbe at a point is defined to be the class in $H^3(B_{x_i}(\epsilon) \setminus \{x_i\}(\epsilon), \mathbb{Z})$ represented by the map $B_{x_i}(\epsilon) \setminus \{x_i\} \to K(Z, 3)$ induced by the gerbe). Here $B_{x_i}(\epsilon)$ is a small ball about x_i. Note that there is a natural generator of $H^3(B_{x_i}(\epsilon) \setminus \{x_i\}(\epsilon), \mathbb{Z})$ given by the orientation of X (or equivalently, the complex structure).

$\mathcal{G}(D)$ *is not a 2-gerbe; it is just a 2-stack.* It's analogous to the line bundle associated to a $\mathbb{G}_m$-torsor; we want to get a 2-gerbe from this object. Namely, we consider the full 2-substack consisting of objects which have order 0 at every point (here again the order is an element of $H^3(B_x(\epsilon) \setminus \{x\}(\epsilon), \mathbb{Z})$). To see that this is an $\mathcal{O}_X^*$-banded 2-gerbe, we must exhibit local isomorphisms with the 2-gerbe of $\mathcal{O}_X^*$-banded gerbes. On an open set U in the complement of the support of D, this is easy; the work is doing this in a neighborhood of a point in D. As in the case of a line bundle, where we exhibit an isomorphism with $\mathcal{O}_X$ by multiplying by a meromorphic function vanishing or blowing up at some point, we must exhibit a gerbe of any order in a neighborhood of a point x, and then "tensor" with that gerbe.

The local situation is as follows—we must exhibit a holomorphic gerbe on the punctured neighborhood of a point, of any order at that point. I'll describe these gerbes on $\mathbb{A}^2 \setminus \{0\}$. To give a holomorphic gerbe, we need to give an open cover on $\mathbb{A}^2 \setminus \{0\}$, a holomorphic "transition line bundle" on double intersections, and "cocycle isomorphisms" on triple intersections. Cover $\mathbb{A}^2 \setminus \{0\}$ by the U_1, U_2, where U_1 is the complement of the x-axis and U_2 the complement of the y-axis. Then $U_1 \cap U_2$ has the homotopy type of $S^1 \times S^1$, and so $H^2(U_1 \cap U_2) = \mathbb{Z}$; the exponential exact sequence shows that $U_1 \cap U_2$ has a $\mathbb{Z}$'s worth of holomorphic line bundles on it. These are our "transition line bundles." There are no interesting triple intersections, so the "cocycle isomorphisms" are vacuous.

This construction shows that our 2-stacks defined above are indeed holomorphic $\mathcal{O}_X^*$-banded gerbes.

2.2. The Riemannian Setting.

I won't say too much about this.

But here's the setup, say where X is a Riemannian three-manifold. Instead of $\mathcal{G}_X^2$ above, we take the stack $\mathcal{G}_X^1$ of meromorphic (almost-everywhere defined) complex line bundles with Yang-Mills connection (recall that this is the condition that the curvature be harmonic).

Then associated to a "0-cycle" $D = \sum a_i[x_i]$ we take the substack $\mathcal{G}(D)$ of line bundles whose Chern classes in a punctured neighborhood of each x_i are at most a_i, and let the associated gerbe consist of the full substack of this consisting of everywhere defined line bundles...

The analytic input is in the fact that this is indeed a gerbe—again this boils down to the construction of line bundles with Yang-Mills connection in a punctured neighborhood of a point, with fixed Chern class. This a bit of (not too bad, elliptic) PDE.

[Comment on "higher generalizations" and algebraicity.]