LIE ALGEBRA PSET I

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1. Lie algebras

A complex Lie algebra is a complex vector space $\mathfrak{g}$ together with a bilinear operation

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that for any $A, B, C \in \mathfrak{g}$, $[-,-]$ satisfies

(1) Anti-commutativity:

$$[A, B] = -[B, A],$$

(2) The Jacobi identity:

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

Some examples:

As a vector space, we define

$$\mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}.$$

That is, $\mathfrak{sl}_n(\mathbb{C})$ consists of the set of $n \times n$ traceless matrices. We set

$$[A, B] = AB - BA.$$

Problem 1. Show that if $A, B \in \mathfrak{sl}_n(\mathbb{C})$, then $[A, B]$ is as well. Check that $\mathfrak{sl}_n(\mathbb{C})$ is a Lie algebra.

Problem 2. Find a few other examples of Lie algebras. (Hint: One may construct an interesting Lie algebra from any non-commutative $\mathbb{C}$-algebra.) Classify all 1-dimensional Lie algebras.

Let us describe $\mathfrak{sl}_2(\mathbb{C})$ in terms of a basis. We set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 3. Show that the set $\{e, f, h\}$ is a basis for $\mathfrak{sl}_2(\mathbb{C})$. Show that

$$[e, f] = h, [h, f] = -2f, [h, e] = 2e,$$

and check that this determines the Lie bracket in general by bilinearity and anti-commutativity.

Definition 1.1. A representation of a Lie algebra $\mathfrak{g}$ is a linear map

$$\rho : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C}) := \mathrm{Mat}_{n\times n}(\mathbb{C})$$

such that

$$\rho([A, B]) = [\rho(A), \rho(B)]$$

for all A, B.

Problem 4. Show that an n-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is the same as a set of three $n \times n$ matrices E, F, H such that

$$[E, F] = H, [H, F] = -2F, [H, E] = 2E.$$

2. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

We now describe explicitly the finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. Let V be a finite-dimensional $\mathbb{C}$-vector space and $E, F, H : V \to V$ linear maps satisfying the commutator relations above. By Problem 4, this is the same as a representation of $\mathfrak{sl}_2(\mathbb{C})$.

Problem 5.* Show that if v is an eigenvector of H with eigenvalue λ, then

$$H(Ev) = (\lambda + 2)Ev$$

and

$$H(Fv) = (\lambda - 2)Fv.$$

Conclude that E, F are nilpotent. (Hint: How many distinct eigenvalues can H have?)

Problem 6.* Let λ be an eigenvalue of H with maximal real part, and let v be a λ-eigenvector. Let N be minimal such that $F^N v = 0$. Show that $Ev = 0$, and conclude that $\mathrm{Span}(v, Fv, F^2 v, \cdots, F^{N-1}v)$ is a subrepresentation of V, i.e. it is preserved by {E,F,V}. Moreover, show that $\lambda = N - 1$.

Problem 7.* Use Problem 6 above to give a description of all finite-dimensional representations V of $\mathfrak{sl}_2(\mathbb{C})$ which are irreducible, up to isomorphism. That is, characterize all V such that if $W \subset V$ is a subrepresentation — a subspace preserved by E, F, H — then either $W = 0$ or $W = V$. Please be explicit, i.e. give a basis and say how E, F, H act on this basis. You should find that there is exactly one irreducible representation of each dimension.

Problem 8.* One may give a more "coordinate-free" description of the irreducible representations. Namely V_N may be viewed as the space of homogeneous polynomials in two variables X, Y of degree $N - 1$. Then E acts via $X\frac{\partial}{\partial Y}$, F via $Y\frac{\partial}{\partial X}$, and H via $X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$. Show that this description agrees with the one you gave in Problem 7.

Problem 9.** Show that any finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct sum of irreducible representations.

Problem 10. Observe that the statement of Problem 9 is not true for general Lie algebras. For example, let $\mathfrak{g}$ be the unique 1-dimensional Lie algebra. Find a representation of $\mathfrak{g}$ which is not a direct sum of irreducible representations.

Problem 11. Use your classification from Problem 7 to show the folloiwng: let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation. Let d_k be the dimension of the generalized k-eigenspace of H acting on V. Then the sequences

$$\{d_k\}_{k \text{ odd}}, \{d_k\}_{k \text{ even}}$$

are both unimodal and symmetric about 0 (i.e. $d_k = d_{-k}$ for all k).