GRAPH ISOMORPHISM AND REPRESENTATION THEORY

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1. The Unimodality Theorem

Recall that $g_{n,k}$ was defined to be the number of unlabeled graphs with n vertices and k edges; we wish to show that for fixed n, the sequence $g_{n,k}$ is unimodal. The idea will to be to construct an $\mathfrak{sl}_2$-representation such that the $g_{n,k}$ appear as dimensions of H-eigenspaces. By Problem 11 of the first problem set, this will suffice.

We fix n, and let W_n be the vector space on the set of *labelled* graphs on the vertices $\{1, \cdots, n\}$. That is, if G_n is the set of labelled graphs on the vertices $\{1, \cdots, N\}$,

$$W_n = \bigoplus_{H \in G_n} \mathbb{C}H.$$

Put another way, W_n is the set of $\mathbb{C}$-valued functions on G_n. The symmetric group S_n acts on G_n by permuting the labels of a labelled graph, and hence acts on W_n. We let $W_{n,k} \subset W_n$ be the subspace spanned by $R \in G_n$ such that R has k edges. Observe that $W_{n,k}$ is a subrepresentation, i.e. it is preserved by the action of S_n.

Problem 1.* Let G be a finite group and V a complex vector space. A *representation* of G is a homomorphism $\rho : G \to GL(V)$. **(a)** Prove Maschke's theorem: If $W \subset V$ is a subrepresentation (i.e. a subspace preserved by the action of G on V), show that there exists another subrepresentation $U \subset V$ so that $V = W \oplus U$. Hint: Pick any projection of V onto W, and average it under the action of G. Let U be the kernel of the G-equivariant projection thus obtained. **(b)** Let

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

Let

$$V_G = V / \operatorname{Span}\{gv - v \mid v \in V, g \in G\}.$$

Show that the natural map

$$V^G \to V_G$$

is an isomorphism. Hint: Use Maschke's theorem.

Problem 2.* Show that

$$g_{n,k} = \dim W_{n,k}^{S_n}.$$

Hint: Use 1(b).

We now define an action of $\mathfrak{sl}_2(\mathbb{C})$ on W_n. Let $e_{i,j} : W \to W$ be the operator

$$e_{i,j} : R \mapsto \begin{cases} R \cup (i,j) & \text{if } (i,j) \notin R \\ 0 & \text{otherwise} \end{cases}$$

i.e. $e_{i,j}$ adds an edge to R between vertices i and j if there isn't one there already, and sends R to 0 otherwise. Let $f_{i,j}$ be the operator

$$f_{i,j} : R \mapsto \begin{cases} R \setminus (i,j) & \text{if } (i,j) \in R \\ 0 & \text{otherwise} \end{cases}$$

i.e. $f_{i,j}$ removes the edge between i and j if such an edge exists, and sends R to 0 otherwise.

Problem 3. Show that if $\{s,t\} \neq \{u,v\}$,

$$[e_{s,t}, f_{u,v}] = 0,$$

and

$$[e_{s,t}, f_{s,t}]R = \begin{cases} R \text{ if } (s,t) \text{ is an edge in } R \\ -R \text{ otherwise.} \end{cases}$$

We set

$$E = \sum_{i<j} e_{i,j}$$

and

$$F = \sum_{i<j} f_{i,j}.$$

Problem 4. Show that for R a labelled graph with n vertices and k edges,

$$[E,F](R) = \left(2k - \binom{n}{2}\right) R.$$

Thus we set $H_k : W_{n,k} \to W_{n,k}$ equal the operator

$$H_k : R \mapsto \left(2k - \binom{n}{2}\right) R,$$

and

$$H = \bigoplus_k H_k.$$

(Here H is an operator $H : W_n \to W_n$.)

Problem 5. We have already checked that $[E,F] = H$. Verify the other two commutation relations for $\mathfrak{sl}_2(\mathbb{C})$, i.e. that

$$[H,F] = -2F, [H,E] = 2E.$$

Problem 6.* Show that the action of $\mathfrak{sl}_2(\mathbb{C})$ on W_n commutes with the action of S_n, and so we get an action of $\mathfrak{sl}_2$ on

$$W_n^{S_n} = \bigoplus_k W_{n,k}^{S_n}.$$

Deduce that the sequence $\{g_{n,k}\}$ (for fixed n and varying k) is unimodal.

2. Supplements

Problem 7.** Let $P_{kl}(n)$ be the number of ways of partitioning n into at most k pieces, each of which has size at most l. Use a similar method to show that for fixed k and l, the sequence

$$P_{kl}(n)$$

is unimodal.

Problem 8. If V, W are representations of a Lie algebra $\mathfrak{g}$, we let $\mathfrak{g}$ act on $V \otimes W$ via the Liebniz rule

$$A(v \otimes w) = (Av) \otimes w + v \otimes (Aw).$$

Show that this is indeed a Lie algebra representation on $V \otimes W$.

Problem 9.* The inclusion $\mathfrak{sl}_2 \to \mathfrak{gl}_2$ gives a canonical 2-dimensional representation of $\mathfrak{sl}_2$; let us call it V. Show that as a $\mathfrak{sl}_2$-representation,

$$W_n \simeq V^{\otimes \binom{n}{2}}.$$

Show that the S_n-action on W_n above may be understood as the natural action of S_n on $V^{\otimes \binom{n}{2}}$ via the natural action of S_n on the set of 2-element sets of a n-element set.