

THE HILBERT SCHEME OF POINTS ON A SURFACE

DANIEL LITT

1. HILBERT AND QUOT SCHEMES AS FUNCTORS

Let Sch/S be the category of locally Noetherian schemes over a Noetherian scheme S ; let $X \rightarrow S$ be an object of Sch/S , where the morphism is of finite type. The functor $\text{Hilb}_{X/S} : \text{Sch}/S^{op} \rightarrow \text{Sets}$ sends a locally Noetherian scheme S' over S to the set

$$\{Z \subset S' \times_S X \mid Z \text{ is closed and } Z \rightarrow S' \text{ is flat and proper}\}.$$

If $f : T' \rightarrow S'$ is a morphism over S , $\text{Hilb}_{X/S}(f)$ sends Z to $T' \times_{S'} Z$, which is closed in $T' \times X$ and flat over T' by the usual properties of base change. Given a polynomial $p(\gamma) \in \mathbb{Q}[\gamma]$ and a line bundle L on X , we have the subfunctor

$$\text{Hilb}_{X/S,p}^L \hookrightarrow \text{Hilb}_X$$

given by

$$\text{Hilb}_{X/S,p}^L(S') = \{Z \subset S' \times_S X \mid Z \text{ is closed and } Z \rightarrow S' \text{ is flat and proper}\}$$

such that the Hilbert polynomial of Z_s computed with respect to L_s is equal to p for each $s \in S'$. (If Z is flat over S the Hilbert polynomial is locally constant, so we have that $\text{Hilb}_{X/S}$ decomposes as a disjoint union of $\text{Hilb}_{X/S,p}^L$ for any p .)

We may generalize this functor slightly. Let E be a coherent sheaf on X with support proper over S . Let $\text{Quot}_{E/X/S}$ be the functor sending S' to the set of equivalence classes of surjections $q : \pi^*(E) \rightarrow F \rightarrow 0$ where $\pi : S' \times_S X \rightarrow X$ is the projection, F is a coherent sheaf on $S' \times_S X$ flat over S' with support proper over S' . (The equivalence relation here is $q \sim q'$ if their kernels are equal.) We have a similar stratification by Hilbert polynomials, as above, with $\text{Quot}_{E/X/S}$ decomposing into $\text{Quot}_{E/X/S,p}^L$. Indeed, the Hilbert functor is a special case of the Quot functor, by setting $E = \mathcal{O}_X$; then closed subschemes identify with ideals of $\mathcal{O}_X$.

We may ask:

In what cases are these functors representable?

Grothendieck shows the following:

Theorem 1. *Let $X \rightarrow S$ be a projective morphism with S Noetherian and L a relatively very ample line bundle on X . Then for any coherent E and polynomial p , the functor $\text{Quot}_{E/X/S,p}^L$ is representable by a projective scheme over S .*

We take this theorem as a given, and proceed to applications using only the functor. We also assume S is an algebraically closed field k .

2. THE TANGENT SPACE TO A HILBERT OR QUOT SCHEME

Let X be a projective scheme over k and E a coherent sheaf on X . Let $q : E \rightarrow F \rightarrow 0$ be a k -point of $\text{Quot}_{E/S/k}$. Then the tangent space at $[q]$ is naturally isomorphic to $\text{Hom}_k(\mathcal{O}_{\text{Quot}_{E/S/k},[q]}, k[\epsilon]/\epsilon^2)$ with

the natural vector space structure. We wish to get a handle on this space. Writing this as the subset of $\text{Hom}(\text{Spec}(k[\epsilon]/\epsilon^2), \text{Quot}_{E/S/k})$ sending the point of $\text{Spec}(k[\epsilon]/\epsilon^2)$ to $[q]$, we may reconsider the problem as follows. Say we have a short exact sequence

$$0 \rightarrow S \rightarrow E \otimes k \rightarrow Q \rightarrow 0$$

(that is, the point $[q]$). We wish to characterize the set of short exact sequences

$$0 \rightarrow S' \rightarrow E \otimes k[\epsilon]/\epsilon^2 \rightarrow Q' \rightarrow 0$$

such that each sheaf is flat over $\text{Spec } k[\epsilon]/\epsilon^2$ and $S' \otimes_{k[\epsilon]/\epsilon^2} k$ agrees with S as a subsheaf of $E \otimes k$.

Lemma 1. *Q' is flat over $k[\epsilon]/\epsilon^2$ if and only if the map $Q' \otimes_{k[\epsilon]/\epsilon^2} (\epsilon) \rightarrow Q'$ is injective.*

Proof. The only if direction is trivial; the if direction follows from considering the long exact sequence for Tor associated to the short exact sequence

$$0 \rightarrow (\epsilon) \rightarrow k[\epsilon]/\epsilon^2 \rightarrow k \rightarrow 0.$$

□

Theorem 2. *Let $e : 0 \rightarrow S \rightarrow E \otimes k \rightarrow Q \rightarrow 0$ be a short exact sequence. The obstruction to an extension of this sequence of the type described above is an element $\text{ob}(e) \in \text{Ext}^1(S, Q \otimes_k (\epsilon))$. If this obstruction is zero, then the set of all extensions is a torsor under $\text{Hom}_k(S, Q \otimes_k (\epsilon))$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & S \otimes_k (\epsilon) & & S & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E \otimes (\epsilon) & \longrightarrow & E \otimes k[\epsilon]/\epsilon^2 & \longrightarrow & E \otimes k \longrightarrow 0 \\
& & \downarrow & & & & \downarrow \\
& & Q \otimes_k (\epsilon) & & & & Q \\
& & \downarrow & & & & \downarrow \\
& & 0 & & & & 0
\end{array}$$

Now let $\alpha : S \otimes_k (\epsilon) \rightarrow E \otimes k[\epsilon]/\epsilon^2$ and $\beta : E \otimes k[\epsilon]/\epsilon^2 \rightarrow Q$ be the maps from the diagram above, and noting that $\beta \circ \alpha = 0$, let $\hat{F} = \ker(\beta)/\text{im}(\alpha)$. Then we have a short exact sequence

$$0 \rightarrow S \otimes_k (\epsilon) \rightarrow \hat{F} \rightarrow Q \rightarrow 0.$$

Let $\text{ob}(e)$ be the class of this sequence in $\text{Ext}^1(S, Q \otimes_k (\epsilon))$. For the rest of the proof it suffices to show that extensions of the type described are in natural bijection with splittings of this sequence.

Indeed, let ξ be a splitting, and let S' be the preimage of $\xi(S)$ in $\ker(\beta)$. Then we have a diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S \otimes_k (\epsilon) & \longrightarrow & S' & \longrightarrow & S \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E \otimes (\epsilon) & \longrightarrow & E \otimes k[\epsilon]/\epsilon^2 & \longrightarrow & E \otimes k \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q \otimes_k (\epsilon) & \longrightarrow & Q' & \longrightarrow & Q \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the middle column is the desired extension (and Q' is flat by the lemma). Similarly, given such a diagram, taking $S'/S \otimes_k (\epsilon)$ gives a subsheaf of $\hat{F}$ isomorphic to S , giving a splitting. Splittings are a torsor under $\mathrm{Hom}_k(Q, S \otimes_k (\epsilon))$ as desired. $\square$

Corollary 1. *The tangent space to $\mathrm{Quot}_{E/X/k}$ at $[q]$ is $\mathrm{Hom}_{\mathcal{O}_X}(\ker(q), Q)$. Letting $E = \mathcal{O}_X$, we have that the tangent space at a point $[Z] \in \mathrm{Hilb}_{X/k}$ is $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$.*

Analyzing the dimension of these vector spaces gives a characterization of local smoothness, as well.

3. HILBERT SCHEME OF POINTS

Let n be a positive integer. We are concerned with the Hilbert scheme $\mathrm{Hilb}_{X/k,n}$, which in some sense (to be made precise soon) parametrizes the set of n -tuples of points in X . As the degree of the Hilbert polynomial of a variety Z is equal to the dimension of the variety, the k -points of $\mathrm{Hilb}_{X/k,n}$ are 0-dimensional subvarieties. In particular, they are 0-dimensional subvarieties satisfying

$$\dim \Gamma(Z, \mathcal{O}_Z) = n.$$

For example, if we let $\{p_1, \dots, p_n\}$ be a collection of n distinct closed points of X , the sheaf $\mathcal{O}_Z$ is the direct sum of the skyscraper sheaves over each point, and thus satisfies this condition. From here on out we will denote the scheme representing $\mathrm{Hilb}_{X/k,n}$ by $X^{[n]}$.

Lemma 2. *If X is connected, $X^{[n]}$ is connected.*

Proof. See FGA explained. (The proof uses the Quot scheme, so we omit it.) $\square$

Theorem 3. *Let X be a irreducible, quasiprojective, and smooth of dimension $1 \leq d \leq 2$. Then $X^{[n]}$ is smooth and irreducible of dimension dn .*

Proof. We first show the smoothness and dimension claims. By generic smoothness, it suffices to show that the tangent space of $X^{[n]}$ is everywhere of dimension dn . We have from our earlier results that at a point $[Z] \in X^{[n]}$ the tangent space is $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$. Consider the short exact sequence

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

and apply the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_Z)$ to get an exact sequence

$$0 \rightarrow \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \rightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \rightarrow \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \rightarrow \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \rightarrow \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z).$$

By e.g. considering the Hilbert polynomial, the first map is an injective map $k^n \rightarrow k^n$ and is thus an isomorphism. Furthermore, $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = H^1(X, \mathcal{O}_Z) = H^1(X, \mathcal{O}_Z(s))$ for $s \in \mathbb{N}$, and is thus zero for large s by the Serre vanishing theorem. So the tangent space is isomorphic to $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$ and it suffices to show that this latter space has dimension equal to nd .

By Serre duality, in the case of a curve we have $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(X, \mathcal{O}_Z \otimes K_X)^\vee = k^n$.

For a surface, we have by flatness that the Euler characteristic

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^2 (-1)^i \dim \mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z)$$

is independent of Z . We have that $\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) = k^n$ by the definition of the Hilbert polynomial and that

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(X, \mathcal{O}_Z \otimes \mathcal{O}_Z \otimes K_X)^\vee = k^n.$$

So the claim is equivalent to the claim that $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 0$.

Let $\mathcal{E}_\bullet \rightarrow \mathcal{O}_Z \rightarrow 0$ be a projective resolution. Then $\sum_i \mathrm{rk}(\mathcal{E}_i) = 0$ and we have that

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^l (-1)^i \dim \mathrm{Hom}(\mathcal{E}_i, \mathcal{O}_Z) = \sum_{i=0}^l (-1)^i \cdot n \cdot \mathrm{rk}(\mathcal{E}_i) = 0$$

by e.g. localizing at the generic point. This does the smoothness and dimension computations.

As for irreducibility, we have by connectedness that any two distinct irreducible components must meet—but at their intersection, the scheme will be singular.

Remark 1. *This is sharp. Consider $(\mathbb{A}^3)^{[4]}$. Then consider a maximal ideal $\mathfrak{m} = (x, y, z)$ and the scheme $\mathcal{O}_Z = \mathcal{O}_p/\mathfrak{m}^2$. Then the tangent space is*

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{Hom}_k(\mathfrak{m}^2, \mathcal{O}_p/\mathfrak{m}^2) = \mathrm{Hom}_k(\mathfrak{m}^2/\mathfrak{m}^3, \mathfrak{m}/\mathfrak{m}^2)$$

by Nakayama. But this has dimension 18; as we will see later, the dimension of the Hilbert scheme $X^{[n]}$ is generically dn when the dimension of X is d , so this scheme cannot be smooth.

□

4. THE HILBERT-CHOW MORPHISM AND CONSEQUENCES.

There is a natural morphism from $X \rightarrow \mathrm{Div}^n(X)$ for X smooth and irreducible. We define this morphism as a morphism of functors. Let S be a scheme and $Z \subset S \times X$ a point of $X^{[n]}(S)$. Then $\det(\mathcal{O}_Z)$ is a relative Cartier divisor of $X \times S/S$; one uses flatness to check that this is compatible with pullbacks. We must check that the Cartier divisor induced is of degree n . We need a lemma

Lemma 3. *Let M be a finitely generated module over a dvr. Then letting $R^n \rightarrow M \rightarrow 0$ be a surjection with n minimal, the kernel is free of rank n and the determinant of the map $R^n \rightarrow R^n$ is independent of the resolution up to R^* . Furthermore, if $0 \rightarrow R^{n'} \rightarrow R^{n'} \rightarrow M \rightarrow 0$ is another resolution of M , the determinant is unchanged.*

Proof. Freeness of the kernel is immediate from properties of dvr's, and the rank condition follows from minimality (the image of the kernel lands in $\mathfrak{m}R^n$, and we use the long exact sequence for $\mathrm{Tor}(k, -)$).

As for independence of the determinant, we may simply add and remove generators one at a time—writing out the matrices, it is clear that they have the same determinant. $\square$

Now to check that the Cartier divisor $\det(\mathcal{O}_Z)$ has degree n , it suffices to check this on Weil divisors. Let v be a point of depth 1; then in a neighborhood of v we may choose a projective resolution of the type described in the lemma. Then the order of vanishing of the determinant of this map at v is the same as the local divisor of the meromorphic function corresponding to this Cartier divisor—summing over all depth 1 points gives the claim. By Yi and Sherry’s results, this gives the map for X a curve.

For higher dimensional X , this construction does not suffice, so we give a construction from FGA explained. We assume X is embedded in $\mathbb{P}^n$, with dual variety $\hat{X} \subset \hat{\mathbb{P}}^n$. In particular, we have $p \mapsto H^p$ (the hyperplane of p) gives an isomorphism $\mathbb{P}^d \simeq \text{Div}^1(\hat{\mathbb{P}}^d)$; summing gives a map $(\mathbb{P}^d)^n \rightarrow \text{Div}^n(\hat{\mathbb{P}}^d)$. The map is S^n -invariant, so there is a natural map $(\mathbb{P}^d)^{(n)} \rightarrow \text{Div}^n(\hat{\mathbb{P}}^d)$. The map is an isomorphism (set-theoretically) into its image; we may also identify $X^{(n)}$ (set-theoretically) with its image.

Let $H \subset \mathbb{P}^d \times \hat{\mathbb{P}}^d$ be the incidence correspondence, with $p, \hat{p}$ the projections. Let Z be a point of $(\mathbb{P}^d)^{[n]}(S)$, and let $Z^* = (\hat{p} \times \text{id}_S)^{-1}(Z) \subset H \times S$; let $\mathcal{F} = (\hat{p} \times \text{id}_S)_*(\mathcal{O}_{Z^*})$ over $\hat{\mathbb{P}}^d \times S$ which is flat over S . Then the determinant is a Cartier divisor on $\hat{\mathbb{P}}^d \times S$ flat over S ; by the lemma it is of degree n and lands in the set-theoretic image of $X^{(n)}$. So we have a map

$$X_{red}^{[n]} \rightarrow X^{(n)}$$

which is in fact surjective.

Note that $X^{(n)}$ has a stratification indexed by partitions of n —namely, the points the subset indexed by ν consist of n -tuples such that the multiplicities of each point give the partition ν . Letting $\nu = (1, 1, 1, \dots, 1)$ we see that this is the image of $X^n - \Delta$, upon which S_n acts freely, and thus has dimension nd . Here the Hilbert-Chow map is an isomorphism, again giving a computation of the generic dimension of $X^{[n]}$.

5. AN EXAMPLE: $(\mathbb{A}^2)^{[n]}$

Theorem 4.

$$(\mathbb{A}^2)^{[n]} = \{(B_1, B_2, i) \in \text{End}(k^n) \times \text{End}(k^n) \times k^n \mid [B_1, B_2] = 0 \text{ and } \forall S \subsetneq k^n \text{ s.t. } i \in S, B_\alpha(S) \not\subset S\} / GL_n(k)$$

where the action of $GL_n(k)$ is given by conjugation on the first two coordinates, and the standard action on the last.

Proof. First, we construct a map from left to right. Points of $(\mathbb{A}^2)^{[n]}$ are ideals I of $A = k[b_1, b_2]$ such that $\dim_k A/I = n$. Viewing A/I as (non-canonically) isomorphic to k^n , we set B_1 equal to the action of b_1 and B_2 equal to the action of b_2 , and let $i = 1 \in A/I$. All the conditions clearly hold. It is clear that any two choices here are related by an action of $GL_n(k)$.

For the map in the other direction, let $\phi_{B_1, B_2, i} : k[b_1, b_2] \rightarrow k^n$ be given by $f \mapsto f(B_1, B_2)i$. Then the image S of $\phi_{B_1, B_2, i}$ satisfies $B_1(S) \subset S, B_2(S) \subset S$ and is thus all of k^n , so the kernel is an ideal of codimension n ; we send (B_1, B_2, i) to the kernel of $\phi_{B_1, B_2, i}$. The kernel is invariant under the action of $GL_n(k)$, completing the proof. $\square$

Consider a point (B_1, B_2, i) in this set; we may ask for a more explicit description of the ideal it represents. If B_1, B_2 each have distinct eigenvalues (which are guaranteed to be non-zero), we may simultaneously diagonalize B_1, B_2 and set $i = (1, \dots, 1)$. Let λ_i be the eigenvalues of B_1 and μ_i the eigenvalues of B_2 . Then the ideal consists of those f such that $f(\lambda_i, \mu_i) = 0$ for all i —that is, it represents n distinct points.

Indeed, in general, the Hilbert-Chow morphism sends (B_1, B_2, i) , after upper triangulization, to $\{(\lambda_i, \mu_i)\}_i$, where λ_i, μ_i are the eigenvalues of B_1, B_2 .