HOMOTOPICAL ENHANCEMENTS OF CYCLE CLASS MAPS

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1. Introduction and Motivation

This work is part of an ongoing attempt to understand the Dold-Thom theorem and its algebro-geometric and arithmetic analogues. The ultimate goal is an analytic description of spaces of cycles algebraically equivalent to 0.

Recall that if X is a connected, finite CW complex, with base point x_0, then

$$\mathrm{Sym}^\infty(X) := \varinjlim(X \xrightarrow{+x_0} \mathrm{Sym}^2(X) \xrightarrow{+x_0} \mathrm{Sym}^3(X) \longrightarrow \cdots).$$

Theorem 1 (Dold-Thom)**.**

$$\mathrm{Sym}^\infty(X) \simeq_{wk} \prod_{i>0} K(H_i(X, \mathbb{Z}), i).$$

The more refined analogues I have in mind are:

Theorem 2 (Dold-Thom + Poincaré Duality)**.** *Let X be a compact oriented manifold of dimension n. Then there is a natural map*

$$\coprod_{n\geq 0} \mathrm{Sym}^n(X) \to \mathrm{Map}(X, B^n\mathbb{Z}).$$

(1) *This map is a group completion (e.g. the induced map*

$$\Omega B\left(\coprod_{n\geq 0} \mathrm{Sym}^n(X)\right) \to \mathrm{Map}(X, B^n\mathbb{Z})$$

is a weak equivalence.)

(2) *If X is connected, the induced maps $\mathrm{Sym}^n(X) \to \mathrm{Map}(X, B^n\mathbb{Z})_n$ become arbitrarily highly connected as $n \to \infty$, where $\mathrm{Map}(X, B^n\mathbb{Z})_n$ is the component corresponding to the cohomology class $n[X]$.*

Theorem 3 (Lefschetz $(1,1)$ + Serre Vanishing)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Then there is a natural map*

$$\mathrm{Hilb}^{\mathrm{codim}\,1}(X) \xrightarrow{[\mathcal{I}^\vee]} \underline{\mathrm{Hom}}(X, B\mathbb{G}_m)$$

induced by dualizing the ideal sheaf of the universal family over Hilb*.*

(1) *This map is a group completion.*

(2) *If $\alpha \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is an ample class, then the induced maps*

$$\mathrm{Hilb}(X, n\alpha) \to \underline{\mathrm{Hom}}(X, B\mathbb{G}_m)_{n\alpha}$$

are highly connected.

In particular, the homotopy type of $\Omega B\,\mathrm{Hilb}^{\mathrm{codim}\,1}(X)$ is

$$K(H^{1,1}(X, \mathbb{Z}), 0) \times K(H^1(X, \mathbb{Z}), 1) \times K(H^0(X, \mathbb{Z}), 0).$$

Remark 1. *For smooth projective curves of genus g, these two statements are the same: if $n > 2g-2$, $\mathrm{Sym}^n(X) \to \mathrm{Pic}^n(X)$ is a $\mathbb{P}^{n-g}$ bundle inducing an isomorphism on π_1 if $n \gg 0$. This sort of algebro-geometric analogue of the product decomposition from the Dold-Thom theorem is what this talk is aimed at emulating.*

Remark 2. *The map $\mathrm{Hilb}^{\mathrm{codim}\,1}(X) \to \underline{\mathrm{Hom}}(X, B\mathbb{G}_m)$ factors through $\underline{\mathrm{Hom}}(X, \mathbb{A}^1/\mathbb{G}_m)$ as an open substack.*

Remark 3. *These sorts of considerations are what motivate the definition of* Lawson homology*.*

Remark 4. *Understanding the geometry and cohomology of Chow and Hilbert schemes is important—for example, rationally connected subvarieties of Chow varieties exactly tell us about rational equivalence of cycles; on the other hand, understanding image of the natural map*

$$H^*(\mathrm{Chow}^p(X), \mathbb{Z}) \to H^{*-2p}(X, \mathbb{Z})$$

would allow one to show that the algebraic part of the intermediate Jacobian of X is defined over a finite extension of the field of definition of X.

Let me sketch the proof of Theorem 3(2), which will presage some future arguments; (1) follows by applying the group completion theorem.

Proof of Theorem 3(2). Consider the a fiber square

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Hilb}(X, n\alpha) \\ \big\downarrow i & & \big\downarrow [\mathcal{I}^\vee] \\ S & \xrightarrow{\mathcal{L}} & \underline{\mathrm{Hom}}(X, B\mathbb{G}_m)_{n\alpha} \end{array}$$

where the bottom map is induced by a line bundle $\mathcal{L}$ on $S \times X$, (which fiberwise has chern class $n\alpha$, and in particular may be taken to be very positive fiberwise). It will suffice to show that i is highly connected.

Let us identify H. A map $T \to H$ is given by a map $r : T \to S$, a relative effective Cartier divisor Z on $T \times X$ which fiberwise has class $n\alpha$, and an isomorphism between $\mathcal{I}_Z^\vee$ and $(r \times X)^*\mathcal{L}$. I claim that for $n \gg 0$ this is $\mathrm{Tot}_X(\pi_*\mathcal{L}) \setminus \{0\}$. This is a $\mathbb{A}^n \setminus \{0\}$ bundle and thus highly connected as desired [flesh this out]. ☐

Most of the rest of this talk will be spent constructing categories which contain both e.g. complex analytic spaces and objects like $B^n\mathbb{Z}$, $B^n\mathbb{G}_m$, or $B^n\mathbb{Z}_D(d)$, as recipients for these sorts of cycle class maps.

The rest of this talk will mostly be technical constructions, so let me briefly flesh out the sort of applications I have in mind, and the properties the objects we want should have. We will end up constructing maps

$$\mathrm{Chow}^p(X) \to \underline{\mathrm{Sect}}(X, B^{2n-2p}\mathbb{Z}_D(d)) \to \underline{\mathrm{Hom}}(X, B^{2n-2p-1}\mathbb{G}_m) \to \underline{\mathrm{Hom}}(X, B^{2n-2p}\mathbb{Z})$$

such that

(1) After applying π_0, these will be precisely cycle class maps.
(2) $\underline{\mathrm{Hom}}(X, B^{2n-2p}\mathbb{G}_m)$ etc. has a universal property.
(3) $X \times \underline{\mathrm{Hom}}(X, B^{2n-2p}\mathbb{G}_m)$ etc. has a universal family over it.
(4) $\mathrm{Chow}^p(X)$ can be recovered as some kind of "space of sections" to this universal family.

This sort of thing (in e.g. the setting of configuration spaces) should be thought of an analogue of "scanning" in algebraic topology. The goal is to e.g. imitate the proof above for other cycle spaces (e.g. zero-cycles Albanese equivalent to some fixed zero cycle).

Theorem 4 (Expected Theorem)**.** *Let X be a smooth, connected, complex projective variety. Let $x_i \in X$ be a sequence of points. Let*

$$\pi : \sqcup \mathrm{Sym}^n(X) \to H^{2d}(X, \mathbb{Z}_D(d))$$

be the cycle class map, and let $X_n = \pi^{-1}(\pi(x_1 + x_2 + \cdots + x_n))$. Then the X_n stabilize homotopically.

As a corollary of the machinery we'll introduce, we'll also cheaply obtain things like

Corollary 1. *Let $K \subset X \times T$ be a family of codimension p cycles. Then the induced map $AJ : T \to \mathcal{J}^p(X)$ is holomorphic.*

2. Simplicial Presheaves

Unfortunately, I now have to give some background on simplicial pre sheaves and homotopy sheaves.

Definition 1. *The simplex category* Δ *is the category whose objects are non-empty totally ordered finite sets*

$$\mathbf{n} := 0 \to 1 \to \cdots \to n$$

and whose morphisms are order-preserving maps. A simplicial set is a (covariant) functor $\Delta^{op} \to \mathrm{Sets}$.

Example 1. *If* $\mathcal{C}$ *is a category, we may associate to it its nerve* $N(\mathcal{C})$*, where*

$$N(\mathcal{C})[n] = \mathrm{Fun}([n], \mathcal{C}).$$

Simplicial sets are a combinatorial model for topological spaces. In particular, there is a functor

$$|\cdot| : sSet \to Top$$

given by

$$X \mapsto \mathrm{colim}\left(\coprod_{[n]\to[k]} X[k] \times \Delta^n \rightrightarrows \coprod_{[n]} X[n] \times \Delta^n\right).$$

Here Δ^n is the simplex $\{(x_0, \ldots, x_n) \subset \mathbb{R}^n \mid x_i > 0, \sum x_i = 1\}$. If we restrict to the category of compactly-generated Hausdorff spaces, this is left adjoint to the functor

$$S \mapsto ([n] \mapsto \mathrm{Hom}(\Delta^n, S)).$$

This allows us to define the homotopy groups of a simplicial set, via

$$\pi_n(X_\bullet) := \pi_n(|X|).$$

The sense in which simplicial sets are a good model for topological spaces is: if we define weak equivalences to be maps f so that $\pi_n(f)$ is an isomorphism for all n, then $sSet[(\text{Weak Equivalences})^{-1}] \to CGHaus[(\text{Weak Equivalences})^{-1}]$ is an equivalence of categories. (The machinery of model categories is designed to (1) deal with set-theoretic difficulties in the constructions of these localized categories, and (2) perform computations in these categories, among other things.)

Remark 5. *Note that the definition of geometric realization makes sense if* X *is a simplicial space or bi-simplicial set; there is also a geometric realization functor from bi-simplicial sets to simplicial sets, where we replace* Δ^n *with* $\Delta[n]$.

We'll also need the notion of homotopy limits and colimits. An ordinary limit or colimit has a universal mapping property, defined by the condition that a map in or out of the limit or colimit is the same as a map to or from the diagram in question. A homotopy colimit will be the same, replacing "a map to or from the diagram in question" with "a map to or from the diagram in question, up to higher coherent homotopies." I'll give a construction of homotopy colimits—the construction of homotopy limits will be dual to this. Namely, suppose we have a diagram of spaces (or simplicial sets, or whatever) $F : I \to sSet$. Then

$$\mathrm{hocolim}_I\, F(i) := |N(F)|$$

where $N(F)$ is the simplicial set

$$N(F)[n] = \coprod_{i_n \to i_{n-1} \to \cdots \to i_0} F(i_n).$$

[Draw picture of homotopy pushout.]

Just to give an example, let me define the homotopy type of a category fibered over complex-analytic spaces. Namely let $F : \mathcal{C} \to \mathrm{CAS}$ be such a category. Then the homotopy type is defined to be

$$\mathrm{hocolim}_{X\in\mathcal{C}}\, F(X)^{top}.$$

We will work with the category sPreCAS of simplicial presheaves on the category of complex-analytic spaces. We can equivalently think of this as $\mathrm{PreCAS}^{\Delta^{op}}$, or as $[\mathrm{CAS}, \mathrm{Sets}^{\Delta^{op}}]$.

Example 2. *If* X *is a complex-analytic space, then* cX *is the simplicial presheaf given by setting* $cX(Y)([n]) = \mathrm{Hom}_{\mathrm{CAS}}(Y, X)$*. This is the constant simplicial presheaf (actually a sheaf) associated to* X*. If* $X_\bullet$ *is a simplicial complex-analytic space (e.g. a hypercover) then* $Y \mapsto ([n] \mapsto \mathrm{Hom}(Y, X[n])$ *is a simplicial presheaf.*

Example 3. *Dold and Kan constructed an equivalence of categories between simplicial abelian groups and non-negatively graded chain complexes of abelian groups,*

$$DK : \mathrm{Ch}^+(\mathrm{Ab}) \to sAb.$$

Thus if $\mathcal{F}^\bullet$ is a non-negatively graded chain complex of presheaves of abelian groups on CAS*, one obtains a simplicial presheaf $DK(\mathcal{F}^\bullet)$.*

Let me be precise about what this correspondence actually is. If $X_\bullet$ is a simplicial abelian group, the normalized chain complex associated to it, $N(X_\bullet)$ has in degree n the group

$$\bigcap_{1\le i\le n} \ker d_i : X_n \to X_{n-1},$$

with d_0 as the differential. Conversely, if $A_\bullet$ is a chain complex, we associate to it the chain complex $DK(A)$, with

$$DK(A_\bullet)[n] = \mathrm{Hom}(N(\mathbb{Z}\Delta[n]), A_\bullet).$$

This construction has the property that $\pi_i(DK(A_\bullet)) = H_i(A_\bullet)$.

Simplicial presheaves have two useful notions of homotopy type. First, there one may associate a simplicial set or topological space to any simplicial presheaf, in analogy with the construction of the homotopy type of a stack above. Namely, to any simplicial presheaf F we may associate a map of simplicial sets $F' : X \to N(\mathrm{CAS})$, where $X = \mathrm{hocolim}_{\mathrm{CAS}} F$ (this is supposed to be the analogue of viewing F as a fibered category). Now, the homotopy type of F is defined to be

$$\mathrm{hocolim}_X F'(-)^{top},$$

e.g. the geometric realization of the simplicial set whose n-th level is

$$\coprod_{f\in \mathrm{Hom}(\Delta[n],X)} f(n)^{top}.$$

The other useful notion of homotopy type of a simplicial presheaf is given by object-wise weak equivalence. In particular, from simplicial presheaf F, we obtain a presheaf given by $\pi_n(F)$. The former notion is like forgetting the complex structure and obtaining a topological space; the latter notion is geometric. For example, if X is a complex-analytic space, $\pi_0(cX)$ remembers the functor of points of X.

Example 4. *If X is projective, the sheafification of $\pi_0(\mathrm{Hom}(X, B\mathbb{G}_m))$ is precisely the functor of points of* $\mathrm{Pic}(X)$*. (Proof: See-saw principle.)*

3. Homotopy Sheaves

Many of the ideas in this section are due to, among others: Jardine, Goerss, Dugger, Morel, Voevodsky.

We'll also need the machinery of "homotopy sheaves"; the prototypical examples are (1) stacks, and (2) complexes of injective sheaves. Suppose $\mathcal{C}$ is a category with a Grothendieck topology. Recall that a sheaf of sets on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathrm{Sets}$ so that if U_α is a covering of X, the sequence

$$F(X) \to \prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta})$$

is an equalizer diagram. We may write this condition equivalently as follows. Suppose $U = \sqcup_\alpha U_\alpha \to X$ is a covering. Then there is a natural simplicial set

$$U_\bullet : \cdots \rightrightarrows U \times_X U \times_X U \rightrightarrows U \times_X U \rightrightarrows U.$$

For the purposes of this talk, a hypercover will be a simplicial set $U'_\bullet$ so that the natural map $U'_\bullet \to (U_0)_\bullet$ is a level-wise covering map. Then a sheaf is, as before, a functor $\mathcal{C}^{op} \to \mathrm{Sets}$ which sends disjoint unions to products and satisfies

$$F(X) = \lim F(U'_\bullet)$$

for any hyper cover $U'_\bullet$ of X.

The idea of a homotopy sheaf is analogous—instead of the above requirement, we ask that the natural map

$$F(X) \to \mathrm{holim}\, F(U'_\bullet)$$

is a weak equivalence, for any hypercover $U'_\bullet$ of X.

Example 5. *Suppose $U_\bullet$ is a cover given by two open subsets U, V, and $\mathcal{F}^\bullet$ is a chain complex of injective sheaves. Then the homotopy sheaf condition for $DK(\mathcal{F}^\bullet)$ is the fact that*

$$\mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$$

is acyclic.

There's a lot of machinery we would like to have for homotopy sheaves (e.g. sheafification), in analogy with the machinery we have for honest sheaves. One way of obtaining this machinery is via the Cech model structure on the category of simplicial presheaves.

The category of simplicial presheaves on a category $\mathcal{C}$ has a (actually two) natural model structures, inherited from the model structures on sSet. One remark is that both of these model structures are *cellular*—I won't say what that means, but the upshot is that we may formally declare any set of maps we'd like to be weak equivalences, and alter the model structure to accommodate this fact (Hirschorn).

I'll use the Heller model structure on sPreCAS; in this model structure, weak equivalences are defined object-wise, as are cofibrations; fibrations are determined by the usual lifting property. (Indeed, this is a simplicial model category.) Now, we localize at the hypercoverings $U_\bullet \to cX$. A homotopy sheaf is precisely a fibrant object in this model structure; the fibrant replacement functor gives a "homotopy sheafification functor."

Remark 6. *This model category is a complex-analytic analogue of an intermediate construction from Morel-Voevodsky's work on $\mathbb{A}^1$-homotopy theory; their construction additionally localizes at projections $X \times \mathbb{A}^1 \to X$.*

I will use the notation HoCAS to denote the homotopy category associated to this model structure. There are two computational tools we will need:

- (Verdier Hypercover Theorem) Suppose $f \in \mathrm{Hom}_{\mathrm{HoCAS}}(X, Y)$. Then f may be represented by the data of a hypercover $U_\bullet \to X$ and a map $f' : U_\bullet \to Y$.
- (Standard Model Category Fact) Suppose Y is fibrant (i.e., a homotopy sheaf). Then any element of $\mathrm{Hom}_{\mathrm{HoCAS}}(X, Y)$ may be represented by a map $X \to Y$.

The first statement is a generalization of the fact that Cech cohomology computes derived functor cohomology, and the latter is a generalization of the fact that we may compute derived functor cohomology with injective resolutions.

4. Constructions

Let me now describe the constructions promised in the introduction. I'll begin with a motivating construction from topology, which I mentioned without comment in the very beginning of the talk. Namely, I claimed that if M is a closed manifold of dimension n, there is a natural map

$$\mathrm{Sym}^n(M) \to \mathrm{Hom}(M, B^n\mathbb{Z})$$

exhibiting the Dold-Thom theorem. Now, there is a natural map $M \times M \to B^n\mathbb{Z}$ given by the class of the diagonal, which gives a map $M \to \mathrm{Hom}(M, B^n\mathbb{Z})$. But $\mathrm{Hom}(M, B^n\mathbb{Z})$ may be made a (strict) commutative monoid, so this induces a map $\mathrm{Sym}^{\mathrm{n}}(M) \to \mathrm{Hom}(M, B^n\mathbb{Z})$, as desired.

Likewise, the existence of maps

$$T \to \mathrm{Hom}(X, -),$$

given a family of cycles over T, will be equivalent to the existence of cycle classes in $H^*(X \times T, -)$, but unwinding the constructions will give more–e.g. universal families, the functor of points of varieties parametrizing cycles as a space of global sections, etc.

In general, for complexes with $A^\bullet$ cycle class maps (e.g. $\mathbb{Z}, \mathbb{G}_m[1], \mathbb{Z}_D(d)$, with

$$\mathbb{Z}_D(d) := 0 \to \mathbb{Z} \xrightarrow{(2\pi i)^d} \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{d-1}_X \to 0$$

there will by maps

$$T \to \mathrm{Hom}(X, B^{2p}A^\bullet)$$

for any family of codimension p cycles in X over T, where X has dimension n. Let me say what this means. Let $B^{2p}A^\bullet$ be the simplicial presheaf associated to the complex A, homologically graded and beginning in degree $2n-2p$. Then there is a simplicial presheaf $\operatorname{Hom}(X, B^{2n-2p}A^\bullet)$ whose T-points are $\operatorname{Hom}(X \times T, \underline{B^{2n-2p}A^\bullet})$, where $\underline{B^{2n-2p}A^\bullet}$ is a fibrant replacement of $B^{2n-2p}A^\bullet$.

The best target of these cycle class maps will be an avatar for Deligne cohomology. Let X be smooth and projective and T an arbitrary complex-analytic space; let $\mathbb{Z}_D(d)^{X,T}$ be a complex of injectives quasi-isomorphic to

$$\mathbb{Z}_D(d)^{X,T} : 0 \to \mathbb{Z} \xrightarrow{(2\pi i)^d} \mathcal{O}_{X\times T} \to \Omega^1_{X\times T/T} \to \Omega^2_{X\times T/T} \to \cdots \to \Omega^{d-1}_{X\times T/T} \to 0.$$

(With some work we may choose this complex functorially.) Regrade this homologically so that $\mathbb{Z}$ is in degree $2d$ and the differential has degree -1. Let $\mathcal{H}^{2d}(X, \mathbb{Z}_D(d))$ denote the homotopy sheaf associated to this complex. Then

(1) The sheafification of $\pi_0(\mathcal{H}^{2d}(X, \mathbb{Z}_D(d)))$ is the functor of points of the complex-analytic space $H^{2d}(X, \mathbb{Z}_D(d))$ (e.g. the intermediate Jacobian etc.)
(2) There is a natural map $\operatorname{Chow}^d(X) \to \mathcal{H}^{2d}(X, \mathbb{Z}_D(d))$ realizing the cycle class map to Deligne cohomology on π_0.

Remark 7. *This construction has the benefit of being a "higher analytic stack." (The other construction with Deligne cohomology does not have this advantage, because Ω^1 is not representable.)*

I will describe an ingredient of the construction of these sorts of cycle class maps (the map to integral cohomology) using the machinery of homotopy sheaves, and then describe the situation with the universal family etc.

Suppose X is smooth and projective, and $Z \subset X \times T$ is a relative codimension p cycle. We wish to construct a map

$$T \to \operatorname{Hom}(X, B^{2p}\mathbb{Z})$$

in HoCAS. Observe that if Z is smooth, this is fine; we can just consider the map $C_*(T \times X, \mathbb{Z}) \to C_*(T \times X, T \times X \setminus Z, \mathbb{Z})$ (via Thom isomorphism). Otherwise, let Z^{sing} be the singular locus of Z; we will use the fact that

$$H^{2r}(X, \mathbb{Z}) \to H^{2r}(X - Z^{sing}, \mathbb{Z})$$

is an isomorphism. Then as before we have a map $T \to \operatorname{Hom}(X - Z^{sing}, B^{2p}\mathbb{Z})$ We wish to find a factorization through $\operatorname{Hom}(X, B^{2p}\mathbb{Z})$. But there is a fiber sequence

$$\operatorname{Hom}(X, B^{2p}\mathbb{Z}) \to \operatorname{Hom}(X - Z^{sing}, B^{2p}\mathbb{Z}) \to \operatorname{Hom}(X, X - Z^{sing}, B^{2p}\mathbb{Z})$$

which such that the map we've constructed is nullhomotopic at the term on the right, and hence lifts (uniquely up to homotopy, again by Thom isomorphism).

Let me now deal with the universal family stuff. I claim that there is a natural map

$$X \times \operatorname{Hom}(X, B^k A^\bullet) \to B^k A^\bullet,$$

given by evaluation. The universal family $\mathcal{U}$ on $X \times \operatorname{Hom}(X, B^k A^\bullet)$ is the homotopy limit of the diagram

$$\begin{array}{ccc} & & \mathrm{pt} \quad . \\ & & \downarrow \\ X \times \operatorname{Hom}(X, B^k A^\bullet) & \longrightarrow & B^k A^\bullet \end{array}$$

This is a torsor for $B^{k-1}A^\bullet$. Now let $f : X \to Y$ be a map of complex-analytic spaces. I claim that there is a natural functor $f_{p*} : \mathrm{sPreCAS}/X \to \mathrm{sPreCAS}/Y$ which associates to a simplicial presheaf Sover X the presheaf of "sections to S over $f^{-1}(U)$ defined away from (relative) codimension p, and only blowing up negatively." [Don't say this.] This extends to a functor $\mathrm{sPreCAS}/X \to \mathrm{sPreCAS}/Y$ for any simplicial presheaves X, Y. Now here is the main theorem

Theorem 5. *Let $f : X \times \operatorname{Hom}(X, B^k A^\bullet) \to \operatorname{Hom}(X, B^k A^\bullet)$ be the projection, and let X be smooth and projective of dimension n. Then the T-points of $\sqcup \operatorname{Sym}^n(X)$ may be identified with the sheafification of $\pi_0(f_{n*}\mathcal{U}/\operatorname{Hom}(X, B^{k-1}A^\bullet))$.*