INVERSION IS SMOOTH

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We usually define a Lie group as a differentiable manifold that is also a topological group, for which the multiplication and inversion maps are smooth. But it turns out that the smoothness of the inversion map follows from the rest of the definition.

Lemma 1. *The set* $\Delta' = \{(g, g^{-1}) \in G \times G\}$ *is an embedded submanifold of* $G \times G$.

Proof. Consider the map $p : G \times G \to G$ given by multiplication; this is a smooth map by assumption and $\Delta' = p^{-1}(e)$. By our lemma on embedded submanifolds from class (or e.g. Spivak Theorem 5-1) it suffices to show that for $x \in \Delta'$, $dp|_x : T_x(G \times G) \to T_e(G)$ is surjective. But this is actually clear; in fact, $dp|_{(a,b)}$ is surjective for all $(a, b) \in G \times G$. To see this, we may for example fix the co-ordinate a; then p is just left multiplication by a, which is a diffeomorphism by assumption. $\square$

Now let $\pi_1, \pi_2 : G \times G \to G$ be the projections from $G \times G$ to the left and right coordinates respectively. Let ι be the embedding $\Delta' \to G \times G$, and let $d : G \to \Delta'$ be the map $g \mapsto (g, g^{-1})$. By e.g. the univeral property of products and embedded submanifolds, π_i and ι are smooth; we claim d is smooth as well. Consider $\pi_1 \circ \iota : \Delta' \to G \times G \to G$ which maps $(g, g^{-1}) \mapsto g$; this is clearly a homeomorphism, and by the inverse function theorem, a diffeomorphism as well. But d is its inverse, and is thus smooth. But then the inversion map is just $\pi_2 \circ \iota \circ d : g \mapsto (g, g^{-1}) \mapsto g^{-1}$, which is the composition of smooth maps and is thus smooth.