A SERRE SYLOW PROBLEM

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This problem was posed to me by Jeremy Booher; it comes from Serre's *Linear Representations of Finite Groups*. The proof I ended up finding was surprisingly constructive, at least to me.

Proposition 1. *Let H be a normal subgroup of a finite group G, and let P_H be a Sylow p-subgroup of G/H. Then there exists a Sylow p-subgroup of G whose image in G/H is P_H.*

Proof. Let P be any Sylow p-subgroup of G. Then its image in G/H is a Sylow p-subgroup P' by counting elements, and is thus conjugate to P_H in G/H by some element $[g] = gH$ (that is, $[g]P'[g]^{-1} = P_H$). Let g be a lift of $[g]$ in G; then gPg^{-1} has image P_H. □

Proposition 2. *Let H be a normal p-group. Then the Sylow p-subgroups of G/H are in bijection with the Sylow p-subgroups of G through the projection $\pi : G \to G/H$.*

Proof. By Proposition 1, the map induced by π is surjective. Assume that two Sylow p-subgroups P and P' map to some Sylow p-subgroup P_H in G/H. Then we show that $A = \{x \mid \pi(x) \in P_H, |x| = p^n\}$ is a subgroup of G; this shows that $P = A = P'$ as P, P' are maximal p-subgroups and A will be a p-subgroup (since every element has order p^n) containing both. By the subgroup criterion, it suffices to show that if $x, y \in A$, then $xy^{-1} \in A$. Clearly $\pi(xy^{-1}) \in P_H$ so we need only check that xy^{-1} has order p^n. But indeed, $\pi(xy^{-1})$ has order $p^{n'}$ as it is in a p-subgroup of G/H, so we have $(xy^{-1})^{n'} \in H$. But H is a p-group, so $(xy^{-1})^{n'}$ has order $p^{n''}$, and thus xy^{-1} has order $p^{n'+n''}$, completing the proof. □

That does the first claim of the second part of Serre's problem.

Proposition 3. *Let H be a central subgroup of G. Then the Sylow p-subgroups of G/H are in bijection with the Sylow p-subgroups of G through the projection $\pi : G \to G/H$.*

Proof. Let S be the (unique, as H is Abelian) Sylow p-subgroup of H; note that S is normal in G. Then we may reduce to the case where $(|H|, p) = 1$ by quotienting G, H by S and applying Proposition 2 above.

Now let $aH \in P_H$; it suffices to show that there is exactly one element of aH with p-th power order. There is at least one such element (call it a'), by Proposition 1. Let $b \in H$. As H is central $|ba'| = |b||a'|$ which is not a p-th power unless $b = 1$, as $(|H|, p) = 1$. This completes the proof. □