

ARITHMETIC REPRESENTATIONS OF FUNDAMENTAL GROUPS I

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ABSTRACT. Let X be a normal algebraic variety over a finitely generated field k of characteristic zero, and let ℓ be a prime. Say that a continuous ℓ -adic representation ρ of $\pi_1^{\text{ét}}(X_{\bar{k}})$ is *arithmetic* if there exists a representation $\tilde{\rho}$ of a finite index subgroup of $\pi_1^{\text{ét}}(X)$, with ρ a subquotient of $\tilde{\rho}|_{\pi_1(X_{\bar{k}})}$. We show that there exists an integer $N = N(X, \ell)$ such that every nontrivial, semisimple arithmetic representation of $\pi_1^{\text{ét}}(X_{\bar{k}})$ is nontrivial mod ℓ^N . As a corollary, we prove that any nontrivial semisimple representation of $\pi_1^{\text{ét}}(X_{\bar{k}})$ which *arises from geometry* is nontrivial mod ℓ^N .

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1. INTRODUCTION

1.1. Statement of main results. The purpose of this paper is to study representations of the étale fundamental group of a variety X over a finitely generated field k — in particular, we wish to understand the restrictions placed on such representations by the action of the absolute Galois group of k on $\pi_1^{\text{ét}}(X_{\bar{k}})$.

Definition 1.1.1. Let k be a field and X a geometrically connected k -variety with a geometric point $\bar{x}$. Then we say that a continuous representation of the geometric étale fundamental group of X

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_{\ell})$$

is *arithmetic* if there exists a finite extension k'/k and a representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_{\ell}),$$

such that ρ is a subquotient of $\tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$.

The main theorem of this paper is:

Theorem 1.1.2. *Let k be a finitely generated field of characteristic zero, and X/k a normal, geometrically connected variety. Let ℓ be a prime. Then there exists $N = N(X, \ell)$ such that any arithmetic representation*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \rightarrow GL_n(\mathbb{Z}_{\ell}),$$

which is trivial mod ℓ^N , is unipotent.

This theorem implies the result of the abstract (that any non-trivial arithmetic representation ρ with $\rho \otimes \mathbb{Q}_\ell$ semisimple is non-trivial mod ℓ^N), because any semisimple unipotent representation is trivial. Note that N is independent of n (the dimension of the representation ρ); to our knowledge this was not expected.

Definition 1.1.3. Let k be a field and X a geometrically connected k -variety with geometric point $\bar{x}$. Then we say that a continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$$

is *geometric* if there exists a smooth proper morphism $\pi : Y \rightarrow X$ and an integer $i \geq 0$ so that ρ appears as a subquotient of the natural monodromy representation

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow GL((R^i \pi_* \mathbb{Z}_\ell)_{\bar{x}}).$$

As a corollary of Theorem 1.1.2, we have:

Corollary 1.1.4. *Let k be any field of characteristic zero, and X/k a normal, geometrically connected variety. Let ℓ be a prime. Then there exists $N = N(X, \ell)$ such that any geometric representation*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \rightarrow GL_n(\mathbb{Z}_\ell),$$

which is trivial mod ℓ^N , is unipotent.

If $k = \mathbb{C}$, we may by standard comparison results replace $\pi_1^{\text{ét}}(X_{\bar{k}})$ with the usual topological fundamental group $\pi_1(X(\mathbb{C})^{\text{an}})$ in Corollary 1.1.4 above. For fixed n (the dimension of ρ) this corollary (but not Theorem 1.1.2) should follow from the main result of [Del87]; the independence of N from n appears to be new and is, we believe, surprising.

In many cases, the invariant $N(X, \ell)$ from Theorem 1.1.2 may be made explicit. For example, if $X = \mathbb{P}_k^1 \setminus \{x_1, \dots, x_m\}$, $N(X, \ell) = 1$ for almost all ℓ ; see Section 4.2 for details. See Section 4.3 for a discussion of the extent to which these results are sharp, and examples of representations which we now know (using Theorem 1.1.2) not to be arithmetic or geometric.

1.2. Theoretical aspects and proofs. The proof of Theorem 1.1.2 is “anabelian” in nature; this is natural as Theorem 1.1.2 is a purely group-theoretic statement about étale fundamental groups of varieties over finitely-generated fields. We think the fact that these anabelian methods have concrete geometric applications (e.g. Corollary 1.1.4) is surprising and interesting.

We now sketch the idea of the proof of Theorem 1.1.2. The proof requires several technical results on arithmetic fundamental groups which are of independent interest. For simplicity we assume X is a smooth, geometrically connected curve over a finitely generated field k ; indeed, one can immediately reduce to this case using an appropriate Lefschetz theorem.

Step 1 (Section 2). Let $x \in X(k)$ be a rational point, and choose an embedding $k \hookrightarrow \bar{k}$; let $\bar{x}$ be the associated geometric point of x . Let $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ be the pro- ℓ completion of the geometric étale fundamental group of X . Let $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ be the $\mathbb{Q}_\ell$ -Mal’cev Hopf algebra associated to $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ (this is an algebra whose continuous representations are the same as the unipotent $\mathbb{Q}_\ell$ -representations of $\pi_1^\ell(X_{\bar{k}}, \bar{x})$), and $W^\bullet$ the weight filtration on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$. Then for any $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1 we construct (Theorem 2.2.4) certain special elements $\sigma_\alpha \in G_k$

which act on $\mathrm{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ via $\alpha^i \cdot \mathrm{Id}$. A key step in the construction of such a σ_α (Lemma 2.2.6) was suggested to the author by Will Sawin [Saw16] after reading an earlier version of this paper, and is related to ideas of Bogomolov [Bog80].

The key input here is a semi-simplicity result (Theorem 2.3.1), which is likely of independent interest. Arguments analogous to those of the proof of Theorem 2.3.1 prove Theorem 2.3.9: that if Y is a smooth variety over $\mathbb{F}_q$, admitting a simple normal crossings compactification, then for $y \in Y(\mathbb{F}_q)$, Frobenius acts semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(Y_{\overline{\mathbb{F}}_q}, \bar{y})]]$.

Step 2 (Section 3). For each real number $r > 0$, we construct certain Galois-stable normed subalgebras

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} \subset \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$$

called “convergent group rings,” which have the following property: If

$$\rho : \pi_1^\ell(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$$

is trivial mod ℓ^m with $r < m$, then there is a natural commutative diagram of continuous ring maps

$$\begin{array}{ccc} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] & \xrightarrow{\rho} & \mathfrak{gl}_n(\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} & \longrightarrow & \mathfrak{gl}_n(\mathbb{Q}_\ell). \end{array}$$

where $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is the group ring of the pro- ℓ group $\pi_1^\ell(X_{\bar{k}}, \bar{x})$, and $\mathfrak{gl}_n(R)$ is the ring of $n \times n$ matrices with entries in R (Proposition 3.1.4). Our second main result on the structure of the Galois action on $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is Theorem 3.2.1, which states that for σ_α as in Step 1, there exists $r_\alpha > 0$ such that $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of σ_α -eigenvectors with dense span, as long as $r > r_\alpha$. Loosely speaking, this means that the denominators of the σ_α -eigenvectors in

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] / \mathcal{I}^n$$

do not grow too quickly in n , where $\mathcal{I}$ is the augmentation ideal.

Step 3 (Section 4). Choose σ_α as in Step 1, with α not a root of unity. Suppose

$$\rho : \pi_1^\ell(X_{\bar{k}}, \bar{x}) \rightarrow GL(V)$$

is an arithmetic representation on a finite free $\mathbb{Z}_\ell$ -module V . Then by a socle argument (Lemma 4.1.1), we may assume that ρ extends to a representation of $\pi_1^{\mathrm{ét}, \ell}(X_{k'}, \bar{x})$ for some k'/k finite. In particular, for m such that $\sigma_\alpha^m \in G_{k'} \subset G_k$, σ_α^m acts on $\mathfrak{gl}(V)$ so that the morphism

$$\rho : \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathfrak{gl}(V)$$

is σ_α^m -equivariant.

Let r_α be as in Step 2, and suppose ρ is trivial mod ℓ^n for some $n > r_\alpha$; choose r with $r_\alpha < r < n$. Thus by Step 2, we obtain a σ_α^m -equivariant map

$$\tilde{\rho} : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} \rightarrow \mathfrak{gl}(V \otimes \mathbb{Q}_\ell).$$

But $\mathfrak{gl}(V \otimes \mathbb{Q}_\ell)$ is a finite-dimensional vector space; thus the action of σ_α^m on $\mathfrak{gl}(V \otimes \mathbb{Q}_\ell)$ has only finitely many eigenvalues. But by Step 2, and our choice of σ_α ,

this implies that for $N \gg 0$,

$$\tilde{\rho}(W^{-N} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}) = 0,$$

where again $W^\bullet$ denotes the weight filtration. It is not hard to see that the W -adic topology and the $\mathcal{I}$ -adic topology on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ agree, where $\mathcal{I}$ is the augmentation ideal. Hence for some $N' \gg 0$,

$$\tilde{\rho}(\mathcal{I}^{N'} \cap \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}) = 0,$$

which implies ρ is unipotent.

1.3. Comparison to existing results. This work was motivated by the geometric torsion conjecture (see e.g. [Cad12]), but is of a rather different nature than most previous results. To our knowledge, most prior results along these lines employ complex-analytic techniques, which began with Nadel [Nad89] and were built on by Noguchi [Nog91] and Hwang-To [HT06]. There has been a recent flurry of interest in this subject, notably two recent beautiful papers by Bakker-Tsimerman [BT16, BT15], which alerted the author to this subject. See also the paper of Brunebarbe [Bru16].

These beautiful results all use the hyperbolicity of $\mathcal{A}_{g,n}$ (the moduli space of principally polarized Abelian varieties with full level n structure) to obstruct maps from curves of low gonality (of course, this description completely elides the difficult and intricate arguments in those papers). The paper [HT06] also proves similar results for maps into other locally symmetric spaces.

The papers [Nad89, Nog91, HT06] together show that there exists an integer $N = N(g, d)$ such that if A is a g -dimensional Abelian variety over a curve $X/\mathbb{C}$ with gonality d , then A cannot have full N -torsion unless it is a constant Abelian scheme. The main deficiency of our result in comparison to these is that we do not give any uniformity in the gonality of X . For example, if $X = \mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$, our results (for example Theorem 4.2.1) depend on the cross-ratios of the x_i . Of course Example 4.3.2 shows that such a dependence is necessary.

Moreover, our result is uniform in g , whereas the best existing results (to our knowledge) are at least quadratic in g . Thus for any given X , our results improve on those in the literature for g large. To our knowledge, this sort of uniformity in g was not previously expected, and is quite interesting. Moreover in many cases our result is sharp. See Section 4.3 for further remarks along these lines.

Our results also hold for arbitrary representations which arise from geometry, rather than just those of weight 1 (e.g. those that arise from Abelian varieties).

Finally, to our knowledge, Theorem 4.2.1 is the first result (apart from [Poo07], which bounds torsion of elliptic curves over function fields) along these lines which works in positive characteristic.

More arithmetic work in this subject has been done by Abramovich-Madapusi Pera-Varilly-Alvarado [AMV16], and Abramovich-Varilly-Alvarado [AV16], which relate the geometric versions of the torsion conjecture to the arithmetic torsion conjecture, assuming various standard conjectures relating hyperbolicity and rational points. Cadoret and Cadoret-Tamagawa have also proven beautiful related arithmetic results (see e.g. [Cad12, CT16, CT12, CT13]). See also e.g. Ellenberg, Hall, and Kowalski's beautiful paper [EHK12].

The technical workhorse of this paper is a study of the action of the Galois group of a finitely generated field k on the geometric fundamental group of a variety X/k . We now compare our results on this subject to those in the literature.

Deligne [Del89, Section 19] studies the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a certain $\mathbb{Z}_\ell$ -Lie algebra associated to a metabelian quotient of $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. In particular, he shows that certain “polylogarithmic” extension classes are torsion, with order given by the valuations of special values of the Riemann zeta function at negative integers. Our results (in Section 3) are much blunter than these. On the other hand, we do give results for the entire fundamental group, rather than a (finite rank) metabelian quotient. Thus we are able to study *integral, non-unipotent* aspects of the representation theory of fundamental groups.

Certain aspects of this work are also intimately related to the so-called ℓ -adic iterated integrals of Wojtkowiak [Woj04, Woj05a, Woj05b, Woj09, Woj12]. In Section 3, we bound certain “integral ℓ -adic periods”; these are related to Wojtkowiak’s ℓ -adic multiple polylogarithms. They are also ℓ -adic analogues of Furusho’s p -adic multiple zeta values [Fur04, Fur07].

One may also view this work as an application of anabelian geometry to Diophantine questions (in particular, about function-field valued points of $\mathcal{A}_{g,n}$ and other period domains). There is a tradition of such applications—see e.g. Kim’s beautiful paper [Kim05] and work of Wickelgren [Wic12b, Wic12a], for example. In particular, we believe our “integral ℓ -adic periods” to be related to Wickelgren’s work on Massey products.

We believe the main contributions of this paper to be:

- (1) the introduction of the rings $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$, and
- (2) the study of the action of G_k , for k a finitely generated field, on these rings.

We also believe the applications of anabelian methods to questions about monodromy is interesting in and of itself — the only antecedent of which we are aware is Grothendieck’s proof of the quasi-unipotent monodromy theorem [ST68, Appendix].

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2. PRELIMINARIES ON FUNDAMENTAL GROUPS AND SOME ASSOCIATED RINGS

2.1. Basic definitions. Let k be a finitely-generated field of characteristic zero, and $\bar{k}$ an algebraic closure of k . Let X be a smooth, geometrically connected variety over k and $\bar{X}$ a simple normal crossings compactification. Such an $\bar{X}$ exists by Hironaka [Hir64]. Let $D = \bigcup_i D_i = \bar{X} \setminus X$ be the boundary, with D_i the irreducible components of D . Let ℓ be a prime. We fix this notation for the rest of the paper.

If $\bar{x}$ is a geometric point of X , we denote by $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ the geometric étale fundamental group of X , and $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ its pro- ℓ completion. If $\bar{x}_1, \bar{x}_2$ are geometric points of X , we let $\pi_1^{\text{ét}}(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$ denote the torsor of étale paths between $\bar{x}_1, \bar{x}_2$, i.e. the pro-finite set of isomorphisms between the fiber functors associated to $\bar{x}_1, \bar{x}_2$ (see e.g. [SGA03, Collaire 5.7 and the surrounding remarks]). The pro-finite set $\pi_1^{\text{ét}}(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$ is a (left) torsor for $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_1)$; we let $\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$ be the associated torsor for $\pi_1^\ell(X_{\bar{k}}, \bar{x}_1)$. One may easily check that the (right) action of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_2)$ on $\pi_1^{\text{ét}}(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$ descends to an action of $\pi_1^\ell(X_{\bar{k}}; \bar{x}_2)$ on $\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$, making it into a (right) torsor for this latter group as well.

Definition 2.1.1. (1) Let

$$\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] = \varprojlim_{\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2) \twoheadrightarrow H} \mathbb{Z}_\ell[H],$$

where the inverse limit is taken over all (continuous) finite quotients H of $\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$. If $\bar{x} = \bar{x}_1 = \bar{x}_2$, this is the ℓ -adic group ring $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$. Note that $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ is naturally a $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x}_1)]]$ - $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x}_2)]]$ -bimodule.

(2) Let

$$\mathcal{I}(\bar{x}) \subset \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$$

be the augmentation ideal (the kernel of the augmentation map

$$\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathbb{Z}_\ell$$

sending $g \mapsto 1$ for $g \in \pi_1^\ell(X_{\bar{k}}, \bar{x})$). Let

$$\mathcal{I}(\bar{x}_1, \bar{x}_2)^n = \mathcal{I}(\bar{x}_1)^n \cdot \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] = \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] \cdot \mathcal{I}(\bar{x}_2)^n.$$

We call this filtration the $\mathcal{I}$ -adic filtration, and if the basepoints are clear, denote it by $\mathcal{I}^n$. Note that the $\mathcal{I}$ -adic topology on $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ is coarser than the profinite topology.

(3) Let

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] = \varprojlim_n (\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] / \mathcal{I}(\bar{x}_1, \bar{x}_2)^n \otimes \mathbb{Q}_\ell),$$

topologized via the $\mathcal{I}$ -adic topology. If $\bar{x} = \bar{x}_1 = \bar{x}_2$, this is $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$, the $\mathbb{Q}_\ell$ -Mal'cev Hopf algebra of $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ (see e.g. [Qui69, Appendix A]). We abuse notation and again denote the $\mathcal{I}$ -adic filtration on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ inherited from $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ by $\mathcal{I}^n$.

Example 2.1.2. Suppose that $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is a free pro- ℓ group on m generators $\gamma_1, \dots, \gamma_m$ (for example, if X is an affine curve). Then the map $\gamma_i \mapsto T_i + 1$ induces isomorphisms

$$\begin{aligned} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] &\xrightarrow{\sim} \mathbb{Z}_\ell\langle\langle T_1, \dots, T_m \rangle\rangle, \\ \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] &\xrightarrow{\sim} \mathbb{Q}_\ell\langle\langle T_1, \dots, T_m \rangle\rangle, \end{aligned}$$

where $R\langle\langle \dots \rangle\rangle$ denotes the ring of noncommutative power series over R . In both cases, $\mathcal{I}$ is the two-sided ideal generated by $T_1, \dots, T_m$. (See e.g. [Iha86, §1].)

If Y is a k -scheme, we define

$$\begin{aligned} H_1(Y_{\bar{k}}, \mathbb{Z}_\ell) &:= \text{Hom}(H^1(Y_{\bar{k}}, \mathbb{Z}_\ell), \mathbb{Z}_\ell), \\ H_1(Y_{\bar{k}}, \mathbb{Q}_\ell) &:= \text{Hom}(H^1(Y_{\bar{k}}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \end{aligned}$$

where $H^1(Y_{\bar{k}}, \mathbb{Z}_\ell)$ is ℓ -adic cohomology, i.e. the inverse limit of the $\mathbb{Z}/\ell^n\mathbb{Z}$ -étale cohomology. It is well-known that

$$H_1(X_{\bar{k}}, \mathbb{Z}_\ell) \simeq \pi_1^\ell(X_{\bar{k}}, \bar{x})^{\text{ab}}$$

canonically for any geometric point $\bar{x}$ of X .

Proposition 2.1.3. *Let $R = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, and let $\mathcal{I}(\bar{x})$ be the augmentation ideal of $R[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$. Then:*

(1) *The map $g \mapsto g - 1$ induces a (canonical) isomorphism*

$$H_1(X_{\bar{k}}, R) \simeq \pi_1^\ell(X_{\bar{k}}, \bar{x})^{\text{ab}} \otimes_{\mathbb{Z}_\ell} R \xrightarrow{\sim} \mathcal{I}(\bar{x}) / \mathcal{I}(\bar{x})^2.$$

(2) *Composition with any element of $\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$ induces isomorphisms*

$$\mathcal{I}(\bar{x}_1) / \mathcal{I}(\bar{x}_1)^2 \simeq \mathcal{I}(\bar{x}_1, \bar{x}_2) / \mathcal{I}(\bar{x}_1, \bar{x}_2)^2 \simeq \mathcal{I}(\bar{x}_2) / \mathcal{I}(\bar{x}_2)^2.$$

These isomorphisms are independent of the choice of element.

- Proof.* (1) This is a standard fact about pro- ℓ groups.
- (2) The inverse morphism is given by composition with the inverse element, in $\pi_1^\ell(X_{\bar{k}}; \bar{x}_2, \bar{x}_1)$. Independence follows from the fact that if p_1, p_2 are two elements of $\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)$, then $p_1 - p_2 \in \mathcal{J}$, as $p_1 - p_2 = (1 - p_2 p_1^{-1}) \cdot p_1$. So if $x \in \mathcal{J}$, $x \cdot (p_1 - p_2) \in \mathcal{J}^2$, and hence is zero in $\mathcal{J}/\mathcal{J}^2$. $\square$

2.2. The weight filtration. Let $\bar{x}$ be a geometric point of X , and $R = \mathbb{Z}_\ell$ or $R = \mathbb{Q}_\ell$. Let

$$\mathcal{J}(\bar{x}) = \ker(R[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow R[[\pi_1^\ell(\bar{X}_{\bar{k}}, \bar{x})]]),$$

where the morphism above is induced by the open embedding $X \hookrightarrow \bar{X}$. We now define the weight filtration $W^\bullet$ on $R[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$. This is an increasing, multiplicative filtration indexed by nonpositive integers.

Definition 2.2.1. For $R = \mathbb{Q}_\ell$:

- $W^i = \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ for $i \geq 0$;
- $W^{-1} = \mathcal{J}(\bar{x})$; $W^{-2} = \mathcal{J}(\bar{x})^2 + \mathcal{J}$;
- $W^{-i} = \sum_{a+b=i, a, b > 0} W^{-a} \cdot W^{-b}$ for $i > 2$.

For $R = \mathbb{Z}_\ell$, let $\iota : \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ be the natural map, and set $W^i \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] = \iota^{-1}(W^i \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]])$.

Remark 2.2.2. One would obtain the same filtration by defining

$$W^{-i} := \mathcal{J}(\bar{x}) \cdot W^{-i+1} + \mathcal{J}(\bar{x}) \cdot W^{-i+2}$$

but the definition above will make several proofs easier; we will not need the equivalence between these two definitions.

Proposition 2.2.3. $\mathcal{J}^n \subset W^{-n}$ and $W^{-2n-1} \subset \mathcal{J}^n$. In particular, the $W^\bullet$ -adic topology is the same as the $\mathcal{J}$ -adic topology.

Proof. Both inclusions follow by induction on n ; one uses that $\mathcal{J}^2 \subset \mathcal{J}^2 + \mathcal{J} \subset \mathcal{J}$. $\square$

Now let $x \in X(k)$ be a rational point of X , and $\bar{x}$ the associated geometric point (given by our choice of algebraic closure of k), so that $G_k := \text{Gal}(\bar{k}/k)$ acts naturally on $\pi_1^\ell(X_{\bar{k}}, \bar{x})$. The main theorem of this section is:

Theorem 2.2.4. For all $\alpha \in \mathbb{Z}_\ell^\times$ sufficiently close to 1, there exists $\sigma_\alpha \in G_k$ such that, for all i , σ_α acts on $\text{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ via $\alpha^i \cdot \text{Id}$.

Before giving the proof, we need several lemmas.

Lemma 2.2.5. Let F be a finite field, let Y/F be a smooth, geometrically connected variety, and let $\bar{Y}$ be a smooth compactification of Y with simple normal crossings boundary. Then G_F acts semi-simply on the ℓ -adic cohomology group $H^1(Y_{\bar{F}}, \mathbb{Q}_\ell)$. Furthermore, this G_F -representation is mixed of weights 1 and 2, with the weight 1 piece given by the image of the natural map

$$H^1(\bar{Y}_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow H^1(Y_{\bar{F}}, \mathbb{Q}_\ell).$$

Proof. Let $j : Y \rightarrow \bar{Y}$ be the embedding. Let $E_1, \dots, E_n$ be the components of $\bar{Y} \setminus Y$. Then the Leray spectral sequence for Rj_* (see e.g. [Del71, 6.2]) gives

$$0 \rightarrow H^1(\bar{Y}_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow H^1(Y_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow \bigoplus_{i \in \{1, \dots, n\}} H^0(E_{i, \bar{F}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-1) \rightarrow H^2(\bar{Y}_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow \dots$$

Now $H^1(\overline{Y}_{\overline{F}}, \mathbb{Q}_\ell)$ is pure of weight 1 by the Weil conjectures, and G_F acts on it semisimply [Tat66]. On the other hand, let

$$V = \ker \left(\bigoplus_{i \in \{1, \dots, n\}} H^0(E_{i, \overline{F}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-1) \rightarrow H^2(\overline{Y}_{\overline{F}}, \mathbb{Q}_\ell) \right).$$

V is manifestly pure of weight 2, with semisimple G_F -action. But the short exact sequence

$$0 \rightarrow H^1(\overline{Y}_{\overline{F}}, \mathbb{Q}_\ell) \rightarrow H^1(Y_{\overline{F}}, \mathbb{Q}_\ell) \rightarrow V \rightarrow 0$$

splits (canonically), as the first and last term have different weights. $\square$

We now prove the following lemma, suggested to us by Will Sawin after reading an earlier draft of this paper — it is closely analogous to results of Serre, Bogomolov, and Deligne (see e.g. [Bog80, Corollaire 1]). It is an analogue of Theorem 2.2.4, but for $H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$ rather than $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]$.

Lemma 2.2.6. *For all $\alpha \in \mathbb{Q}_\ell^\times$ sufficiently close to 1, there exists $\sigma_\alpha \in G_k$ such that σ_α acts on $\mathrm{gr}_W^{-i} H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$ via multiplication by α^i . Here $W^i H_1(X_{\overline{k}}, \mathbb{Q}_\ell) = H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$ for $i \geq -1$,*

$$W^{-2} H_1(X_{\overline{k}}, \mathbb{Q}_\ell) = \ker(H_1(X_{\overline{k}}, \mathbb{Q}_\ell) \rightarrow H_1(\overline{X}_{\overline{k}}, \mathbb{Q}_\ell)),$$

and $W^i H_1(X_{\overline{k}}, \mathbb{Q}_\ell) = 0$ for $i < -2$.

Proof. The proof proceeds in two steps. First we show that there exists a σ_α as desired in the Zariski-closure of the image of the natural Galois representation

$$\rho : G_k \rightarrow GL(H_1(X_{\overline{k}}, \mathbb{Q}_\ell)).$$

Second, we use the fact that the image of ρ is open in its Zariski-closure to conclude.

Step 1. We first show that for any $\alpha \in \mathbb{Q}_\ell^\times$, there exists an element γ_α of $\overline{\mathrm{im}(\rho)}$ preserving $W^\bullet H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$ and acting on $\mathrm{gr}_W^{-i} H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$ via α^i . Here $\overline{\mathrm{im}(\rho)}$ is the Zariski-closure of the image of ρ .

The filtration $W^\bullet$ in the statement of the lemma agrees with the weight filtration of Deligne (see e.g. [Del71, 6]), so in particular there exist (many) Frobenii γ in G_k such that γ acts with weight i on $\mathrm{gr}_W^i H_1(X_{\overline{k}}, \mathbb{Q}_\ell)$. (Recall that this means that this means that if λ is a generalized eigenvalue of γ acting on gr_W^i , then λ is an algebraic number, and $|\lambda| = q^{-i/2}$ for any embedding of $\mathbb{Q}(\lambda)$ into $\mathbb{C}$.) Choose such a γ .

There is a short exact sequence

$$0 \rightarrow \mathrm{gr}_W^{-2} H_1(X_{\overline{k}}, \mathbb{Q}_\ell) \rightarrow H_1(X_{\overline{k}}, \mathbb{Q}_\ell) \rightarrow \mathrm{gr}_W^{-1} H_1(X_{\overline{k}}, \mathbb{Q}_\ell) \rightarrow 0.$$

Because γ acts on gr_W^{-i} with weight $-i$, there is a canonical γ -equivariant splitting of this sequence, so we have a canonical γ -equivariant isomorphism

$$\mathrm{gr}_W^{-1} H_1(X_{\overline{k}}, L) \oplus \mathrm{gr}_W^{-2} H_1(X_{\overline{k}}, L) \simeq H_1(X_{\overline{k}}, L). \quad (2.2.7)$$

By Lemma 2.2.5, there exists a finite extension L of $\mathbb{Q}_\ell$ so that γ acts diagonalizably on $H_1(X_{\overline{k}}, L)$. Choose a basis $\{e_i\}$ of γ -eigenvectors of $H_1(X_{\overline{k}}, L)$ adapted to splitting 2.2.7, so that $e_1, \dots, e_r$ forms a basis of gr_W^{-2} , and $e_{r+1}, \dots, e_m$ forms a basis of gr_W^{-1} . By the definition of weights, we have

$$\gamma \cdot e_i = \lambda_i e_i,$$

with $|\lambda_i| = q$ if $1 \leq i \leq r$ and $|\lambda_i| = q^{1/2}$ otherwise, where $|\cdot|$ is defined via any embedding of $\mathbb{Q}(\lambda_1, \dots, \lambda_m)$ into $\mathbb{C}$. Now the identity component T of $\{\gamma^n\}_{n \in \mathbb{Z}}$ is a subtorus of the diagonal torus D of $GL(H_1(X_{\bar{k}}, L))$ (in the basis $\{e_i\}$).

Let $X^*(D), X^*(T)$ be the character lattices of D, T respectively; identify $X^*(D) \simeq \mathbb{Z}^m$ via the basis $\{e_i\}$. The inclusion $T \hookrightarrow D$ induces a surjection $X^*(D) \twoheadrightarrow X^*(T)$, with kernel K given by $\underline{a} \in \mathbb{Z}^m$ such that

$$\prod_{i=1}^m \lambda_i^{a_i} = 1.$$

T is precisely the torus cut out by the characters in K , i.e. the subtorus given by diagonal matrices M such that $\chi(M) = 1$ for all $\chi \in K$. But in particular, this holds for the matrices

$$q^a \cdot \text{Id}_{\text{gr}_W^{-1}} \oplus q^{2a} \cdot \text{Id}_{\text{gr}_W^{-2}}, \quad a \in \mathbb{Z}$$

by the multiplicativity of absolute values. Hence these matrices are in T , which thus contains their Zariski-closure, the entire torus T'

$$\alpha \cdot \text{Id}_{\text{gr}_W^{-1}} \oplus \alpha^2 \cdot \text{Id}_{\text{gr}_W^{-2}}, \quad \alpha \in \mathbb{Q}_\ell^\times.$$

As the splitting 2.2.7 is defined over $\mathbb{Q}_\ell$, this torus is in fact in $\overline{\text{im}(\rho)}$, as desired.

Step 2. Now we demonstrate that $\text{im}(\rho)$ is open in $\overline{\text{im}(\rho)}$, and use this fact to conclude the proof of the lemma. In the case k is a number field, this follows from the main result of [Bog80, Théorème 1], once we verify that for all places $\mathfrak{l}$ of k lying over ℓ , the restriction of ρ to the decomposition group at $\mathfrak{l}$ is Hodge-Tate. But in fact this representation is de Rham, hence Hodge-Tate, by the main result of [Kis02]. For k an arbitrary finitely-generated field of characteristic zero, we may reduce to the number field case by the argument of [Ser13, letter to Ribet of 1/1/1981, §1].

To conclude, we observe that $\text{im}(\rho) \cap T'$ is thus open and non-empty, and in particular contains a non-empty open neighborhood of $1 \in T'$. This is exactly what we wanted to prove. $\square$

2.3. Semisimplicity. The purpose of this subsection is to deduce Theorem 2.2.4 from Lemma 2.2.6. Before doing so, we will need a result (which may be of independent interest) proving that certain elements of G_k act semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$.

Theorem 2.3.1. *Let σ_α be as in Lemma 2.2.6, with α not a root of unity; let x be a rational point of X . Then σ_α acts semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n$ for all n .*

The proof of this theorem is rather easy if $H_1(X_{\bar{k}}, \mathbb{Q}_\ell)$ is pure (in this case, the $\mathcal{I}$ -adic filtration splits σ_α -equivariantly for formal reasons), but requires some work in the mixed setting. Before proceeding with the proof of Theorem 2.3.1, we will need some auxiliary definitions and results.

Let $k((t^{1/\infty})) = \bigcup_n k((t^{1/n}))$. Choose an algebraic closure $\overline{k((t^{1/\infty}))}$ and an identification of $\bar{k}$ with the algebraic closure of k in $\overline{k((t^{1/\infty}))}$.

Proposition 2.3.2. *The inclusion $k \hookrightarrow k((t^{1/\infty}))$ induces an isomorphism $G_k \simeq G_{k((t^{1/\infty}))}$.*

Proof. This follows from the algebraic closedness of the field of Puiseux series over an algebraically closed field of characteristic zero. $\square$

Definition 2.3.3. A *rational tangential basepoint* of X is a $k((t))$ -point of X . If x is a rational tangential basepoint of X , we let $\bar{x}$ denote the geometric point of X obtained from our choice of algebraic closure of $k((t^{1/\infty}))$ (which is also an algebraic closure of $k((t))$).

Because of Proposition 2.3.2, a rational tangential basepoint x of X induces a natural action of G_k on $\pi_1^\ell(X_{\bar{k}}, \bar{x})$.

Proposition 2.3.4. *Let x_1, x_2 be rational points or rational tangential basepoints of X , and $\bar{x}_1, \bar{x}_2$ the associated geometric points. Then the G_k -representation on the $\mathbb{Q}_\ell$ -vector space*

$$\mathcal{I}(\bar{x}_1, \bar{x}_2)^n / \mathcal{I}(\bar{x}_1, \bar{x}_2)^{n+1}$$

(where if one or both of the x_i is a rational tangential basepoint, we view this as a G_k -representation via the isomorphism from Proposition 2.3.2) is mixed with weights in $[-2n, -n]$. For σ_α as in Lemma 2.2.6, σ_α acts on $\mathcal{I}(\bar{x}_1, \bar{x}_2)^n / \mathcal{I}(\bar{x}_1, \bar{x}_2)^{n+1}$ semi-simply, with eigenvalues contained in $\{\alpha^n, \alpha^{n+1}, \dots, \alpha^{2n}\}$.

Proof. If $n = 0$ this is trivial; if $n = 1$ it follows from Propositions 2.1.3 and 2.2.5. In general, note that the composition map

$$(\mathcal{I}(\bar{x}_1) / \mathcal{I}(\bar{x}_1)^2)^{\otimes n-1} \otimes \mathcal{I}(\bar{x}_1, \bar{x}_2) / \mathcal{I}(\bar{x}_1, \bar{x}_2)^2 \rightarrow \mathcal{I}(\bar{x}_1, \bar{x}_2)^n / \mathcal{I}(\bar{x}_1, \bar{x}_2)^{n+1}$$

is Galois-equivariant and surjective. As weights are additive in tensor products, this completes the proof. $\square$

Proposition-Construction 2.3.5. *Let σ_α be as in Lemma 2.2.6, with α not a root of unity. Let x_1, x_2 be rational points or rational tangential basepoints of X . Then there exists a unique element $p(\bar{x}_1, \bar{x}_2) \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ characterized by the following two properties:*

- (1) $p(\bar{x}_1, \bar{x}_2)$ is fixed by σ_α , and
- (2) $\epsilon(p(\bar{x}_1, \bar{x}_2)) = 1$, where $\epsilon : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] \rightarrow \mathbb{Q}_\ell$ is the augmentation map.

We refer to $p(\bar{x}_1, \bar{x}_2)$ as the canonical path between $\bar{x}_1, \bar{x}_2$.

Proof. Such $p(\bar{x}_1, \bar{x}_2)$ are in bijection with σ_α -equivariant splittings s of ϵ , by setting $p(\bar{x}_1, \bar{x}_2) = s(1)$. So it is enough to prove that there is a unique σ_α -equivariant splitting of ϵ .

Because $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]]$ is complete with respect to the $\mathcal{I}$ -adic filtration, it is enough to show that the augmentation maps

$$\epsilon_n : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}_1, \bar{x}_2)]] / \mathcal{I}^n \rightarrow \mathbb{Q}_\ell$$

admit unique σ_α -equivariant splittings. But

$$\ker(\epsilon_n) = \mathcal{I} / \mathcal{I}^n,$$

on which σ_α acts with generalized eigenvalues in $\{\alpha, \dots, \alpha^{2n}\}$ by Proposition 2.3.4, which does not contain 1 by assumption. So the result is clear. $\square$

Remark 2.3.6. We have $p(\bar{x}, \bar{x}) = 1$, and by uniqueness $p(\bar{x}_1, \bar{x}_2) \cdot p(\bar{x}_2, \bar{x}_3) = p(\bar{x}_1, \bar{x}_3)$. Hence in particular $p(\bar{x}_1, \bar{x}_2) = p(\bar{x}_2, \bar{x}_1)^{-1}$.

Proof of Theorem 2.3.1. Let r_n be the quotient map $r_n : \mathcal{J}/\mathcal{J}^n \rightarrow \mathcal{J}/\mathcal{J}^2$. First, we argue that it is enough to produce a σ_α -equivariant splitting s_n of the map r_n . Indeed, then s_n induces a surjective, σ_α -equivariant map

$$\bigoplus_{i=0}^{n-1} (\mathcal{J}/\mathcal{J}^2)^{\otimes i} \xrightarrow{\oplus s_n^{\otimes i}} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{J}^n.$$

As σ_α acts semi-simply on the source of this map, it does so on the target as well.

We now construct such an s_n . Let $V_1 \oplus V_2$ denote the splitting of $\mathcal{J}/\mathcal{J}^2 \simeq H_1(X_{\bar{k}}, \mathbb{Q}_\ell)$ into σ_α -eigenspaces, where σ_α acts on V_1 via $\alpha \cdot \text{Id}$ and on V_2 via $\alpha^2 \cdot \text{Id}$. Let $t_i : \mathcal{J}/\mathcal{J}^2 \rightarrow V_i, i = 1, 2$ be the natural quotient maps. It suffices to construct splittings of $t_i \circ r_n$ for each i .

For $i = 1$, note that by Proposition 2.3.4, the generalized eigenvalues of σ_α on $\ker(t_1 \circ r_n)$ are contained in $\{\alpha^2, \alpha^3, \dots, \alpha^{2n-2}\}$; as α does not appear on this list, there is a (unique) σ_α -equivariant splitting.

So we need only construct a σ_α -equivariant splitting of $t_2 \circ r_n$. In this case, there is no “weight” reason for the existence of such a splitting: α^2 may very well appear as an eigenvalue of σ_α acting on $\ker(t_2 \circ r_n)$. Instead, we give a direct construction.

Without loss of generality, we may assume that V_2 is nonzero. We may replace k by a finite extension so that each of the c components D_i of the simple normal crossings divisor $D = \bigcup_i D_i = \bar{X} \setminus X$ is geometrically connected and has a rational point x_i^0 in the smooth locus of D . (We replace σ_α by a power so it lies in the Galois group of this extension of k ; it suffices to prove semisimplicity for this power.) For each i , choose a $k[[t]]$ -point of $\bar{X}$ transverse to D_i , so that the special point of $\text{Spec}(k[[t]])$ is sent to x_i^0 . Let x_i be the associated $k((t))$ -point, $\tilde{x}_i$ the associated $\bar{k}((t))$ -point, and $\bar{x}_i$ the associated $\bar{k}((\bar{t}))$ -point.

Note that the map $\sqcup_i \tilde{x}_i \rightarrow X$ induces a surjection

$$u : \mathbb{Q}_\ell(1)^{\oplus c} \simeq \bigoplus_i H_1(\tilde{x}_i, \mathbb{Q}_\ell) \rightarrow H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \twoheadrightarrow V_2,$$

by e.g. the proof of Lemma 2.2.5. We first argue that it is enough to produce a σ_α -equivariant lift of this map to $\mathcal{J}/\mathcal{J}^n$, i.e. to construct the dotted arrow w below.

$$\begin{array}{ccc} & & \mathbb{Q}_\ell(1)^{\oplus c} \simeq \bigoplus_i H_1(\tilde{x}_i, \mathbb{Q}_\ell) \\ & \nearrow w & \downarrow u \\ \mathcal{J}/\mathcal{J}^n & \xrightarrow{r_n} \mathcal{J}/\mathcal{J}^2 = H_1(X_{\bar{k}}, \mathbb{Q}_\ell) & \xrightarrow{t_2} V_2. \end{array}$$

Indeed, σ_α evidently acts semisimply on $\mathbb{Q}_\ell(1)^{\oplus c}$, so the vertical map u above admits a σ_α -equivariant section s . Composing it with the dotted arrow w will give the desired section to $t_2 \circ r_n$.

We now produce w as desired. Let γ be a generator of the maximal pro- ℓ quotient I of the inertia group of $G_{k((t))}$; the map $x_i \rightarrow X$ induces a map

$$\iota_{x_i} : I \rightarrow \pi_1^\ell(X_{\bar{k}}, \bar{x}_i).$$

Observe that there is a canonical isomorphism $I \simeq H_1(\tilde{x}_i, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell(1)$. Note moreover that the maps $u_i : H_1(\tilde{x}_i, \mathbb{Z}_\ell) \rightarrow V_2$ (the direct summands of u) factor as

$$u_i : H_1(\tilde{x}_i, \mathbb{Z}_\ell) \simeq I \xrightarrow{\iota_{x_i}} \pi_1^\ell(X_{\bar{k}}, \bar{x}_i) \rightarrow \pi_1^\ell(X_{\bar{k}}, \bar{x}_i)^{\text{ab}} \otimes \mathbb{Q}_\ell \simeq H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \twoheadrightarrow V_2.$$

Of course it suffices to construct σ_α -equivariant lifts of u_i to $\mathcal{J}/\mathcal{J}^n$ for each i ; we claim that

$$\gamma \mapsto p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x})$$

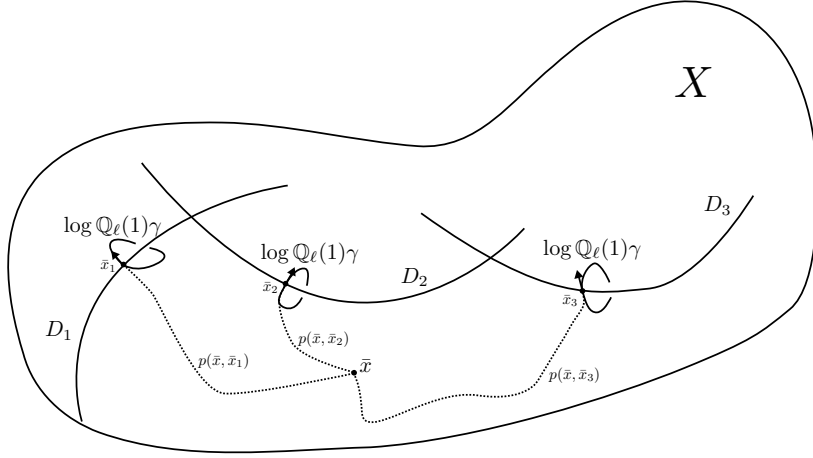


FIGURE 2.3.7. A schematic depiction of the lift of the weight -2 part of $\mathcal{S}(\bar{x})/\mathcal{S}(\bar{x})^2$ to $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$

is such a lift, where $p(\bar{x}, \bar{x}_i)$ is the canonical path from Proposition-Construction 2.3.5. Here $\log(\iota_{x_i}(\gamma))$ is the power series

$$\log(\iota_{x_i}(\gamma)) = \log(1 + (\iota_{x_i}(\gamma) - 1)) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(\iota_{x_i}(\gamma) - 1)^j}{j},$$

which is in fact a finite sum because $(\iota_{x_i}(\gamma) - 1) \in \mathcal{S}$.

We must check that this is (a) a lift, and (b) σ_α -equivariant. To check statement (a), we must show that

$$p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x}) = \iota_{x_i}(\gamma) - 1 \bmod \mathcal{S}^2,$$

by Proposition 2.1.3(1). Because $(\iota_{x_i}(\gamma) - 1) \in \mathcal{S}$, we have

$$\begin{aligned} p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x}) &= p(\bar{x}, \bar{x}_i) \cdot (\iota_{x_i}(\gamma) - 1) \cdot p(\bar{x}_i, \bar{x}) \bmod \mathcal{S}^2 \\ &= \iota_{x_i}(\gamma) - 1 \bmod \mathcal{S}^2, \end{aligned}$$

where the first equality follows from the power series definition of $\log$ and the second from the same argument as in the proof of Proposition 2.1.3(2). To check statement (b), we must verify that

$$\sigma_\alpha(p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x})) = \alpha^2 \cdot p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x}).$$

As V_2 was assumed to be non-zero, and is a quotient of $\mathbb{Q}_\ell(1)^{\oplus c}$, we know that $\alpha^2 = \chi(\sigma_\alpha)$ where $\chi : G_k \rightarrow \mathbb{Z}_\ell^\times$ is the cyclotomic character. Since G_k acts on I via the cyclotomic character, we have

$$\begin{aligned} \sigma_\alpha(p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x})) &= \sigma_\alpha(p(\bar{x}, \bar{x}_i)) \cdot \log(\iota_{x_i}(\gamma^{\chi(\sigma_\alpha)})) \cdot \sigma_\alpha(p(\bar{x}_i, \bar{x})) \\ &= p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)^{\alpha^2}) \cdot p(\bar{x}_i, \bar{x}) \\ &= \alpha^2 \cdot p(\bar{x}, \bar{x}_i) \cdot \log(\iota_{x_i}(\gamma)) \cdot p(\bar{x}_i, \bar{x}) \end{aligned}$$

as desired, where the first equality follows from the fact that composition of paths and ι_{x_i} commute with the G_k -action, the second from the fact that the $p(\bar{x}, \bar{x}_i)$ are σ_α -invariant (by definition), and the last from the identity $\log(y^n) = n \log(y)$. This completes the proof. $\square$

Remark 2.3.8. An essentially identical argument proves:

Theorem 2.3.9. *Let $Y/\mathbb{F}_q$ be a smooth, geometrically connected variety admitting a simple normal crossings compactification, with ℓ prime to q . Let $y \in Y(\mathbb{F}_q)$ be a rational point. Then Frobenius acts semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(Y_{\overline{\mathbb{F}}_q}, \bar{y})]]/\mathcal{I}^n$ for any n .*

The only difference in the proof is that one replaces α^j with a q -Weil number of weight $-j$ whenever necessary.

Proof of Theorem 2.2.4. It suffices to prove the theorem with $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ replaced by $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n$ with the induced W -adic filtration, by Proposition 2.2.3.

Let σ_α be as in Lemma 2.2.6; we claim that this same σ_α also satisfies the conclusions of Theorem 2.2.4. Indeed, by Theorem 2.3.1, we may choose a σ_α -equivariant splitting s of the quotient map $\mathcal{I}/\mathcal{I}^n \rightarrow \mathcal{I}/\mathcal{I}^2$. The induced map

$$\bigoplus_{i=0}^{n-1} H_1(X_{\bar{k}}, \mathbb{Q}_\ell)^{\otimes i} \simeq \bigoplus_{i=0}^{n-1} (\mathcal{I}/\mathcal{I}^2)^{\otimes i} \xrightarrow{\oplus s^{\otimes i}} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n$$

is surjective and σ_α -equivariant, so we are done by the multiplicativity of the $W^\bullet$ -filtration. $\square$

3. INTEGRAL ℓ -ADIC PERIODS AND CONVERGENT GROUP RINGS

3.1. Convergent group rings. In the previous section we constructed (Theorem 2.2.4) certain elements $\sigma_\alpha \in G_k$ acting semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$; in particular, $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ admits a set of σ_α -eigenvectors with dense span (in the $\mathcal{I}$ -adic topology). This is typically not the case for $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$.

Example 3.1.1. Suppose $X = \mathbb{G}_m$, and let x be any k -rational point of X . Then $\pi_1^\ell(X_{\bar{k}}, \bar{x}) = \mathbb{Z}_\ell(1)$. Let γ be a topological generator of $\mathbb{Z}_\ell(1)$. Via the map $\gamma \mapsto T+1$ we have:

- $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \xrightarrow{\sim} \mathbb{Z}_\ell[[T]]$, $\mathcal{I}^n = W^{-2n} = W^{-2n+1} = (T^n)$;
- $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \xrightarrow{\sim} \mathbb{Q}_\ell[[T]]$, $\mathcal{I}^n = W^{-2n} = W^{-2n+1} = (T^n)$.

For $\sigma \in G_k$, we have

$$\sigma(1+T) = (1+T)^{\chi(\sigma)}$$

where $\chi : G_k \rightarrow \mathbb{Z}_\ell^\times$ is the cyclotomic character. The elements

$$(\log(1+T))^n \in \mathbb{Q}_\ell[[T]], n \in \mathbb{Z}_{\geq 0}$$

are σ -eigenvectors with eigenvalue $\chi(\sigma)^n$; their span is evidently dense in the (T) -adic topology, as $(\log(1+T))^n$ has leading term T^n . On the other hand, if $\chi(\sigma) \neq 1$, the only σ -eigenvector in $\mathbb{Z}[[T]]$ is 1.

Observe that the σ -eigenvectors in the example above, $(\log(1+T))^n$, are power series with positive ℓ -adic radius of convergence. The purpose of this section is to generalize this observation to arbitrary X .

Definition 3.1.2. Let $r > 0$ be a positive real number. Let

$$\pi_n : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n$$

be the quotient map, and $v_n : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n \rightarrow \mathbb{Z} \cup \{\infty\}$ the valuation defined by

$$v_n(p) = -\inf\{m \in \mathbb{Z} \mid \ell^m \cdot p \in \text{im}(\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n \rightarrow \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/\mathcal{I}^n)\}.$$

We define the *convergent group ring of radius ℓ^{-r}* to be

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} := \{p \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \mid v_n(\pi_n(p)) + nr \rightarrow \infty \text{ as } n \rightarrow \infty\},$$

topologized via the r -Gauss norm

$$|p|_r := \sup_n \{\ell^{-v_n(\pi_n(p)) - nr}\}.$$

We again use the notation $\mathcal{J}^n, W^{-i}$ to denote the filtrations on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ inherited from $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$.

Loosely speaking, $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ consists of those elements whose denominators do not “grow too quickly” modulo $\mathcal{J}^n$. Note that it is *not* given the subspace topology, but rather the Gauss norm topology, in all that follows.

Example 3.1.3. Suppose $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is a free pro- ℓ group, generated by $\gamma_1, \dots, \gamma_m$ (for example, if X is an affine curve). Then the map $\gamma_i \mapsto 1 + T_i$ induces an isomorphism

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \xrightarrow{\sim} \mathbb{Q}_\ell\langle\langle T_1, \dots, T_m \rangle\rangle,$$

where $\mathbb{Q}_\ell\langle\langle T_1, \dots, T_m \rangle\rangle$ is the ring of non-commutative power series in $T_1, \dots, T_m$ over $\mathbb{Q}_\ell$. The subring $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ consists of those power series $\sum a_I T^I$ such that

$$\lim_{|I| \rightarrow \infty} v_\ell(a_I) + |I|r = \infty.$$

If $m = 1$, this is precisely the set of univariate power series converging on the closed ball of radius ℓ^{-r} . The Gauss norm is given by

$$\left| \sum a_I T^I \right|_r = \sup_I \ell^{-v_\ell(a_I) - |I|r}.$$

In fact $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ is the completion of $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \otimes \mathbb{Q}_\ell$ at the norm $|\cdot|_r$.

The following proposition justifies the terminology *convergent group ring*.

Proposition 3.1.4. *Suppose $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is a finitely-generated free pro- ℓ group, and let $\rho : \pi_1^\ell(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$ be a continuous representation which is trivial mod ℓ^m . Then for any $0 < r < m$, there exists a unique continuous ring homomorphism $\tilde{\rho} : \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} \rightarrow \mathfrak{gl}_n(\mathbb{Q}_\ell)$ making the diagram*

$$\begin{array}{ccc} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] & \xrightarrow{\rho} & \mathfrak{gl}_n(\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} & \xrightarrow{\tilde{\rho}} & \mathfrak{gl}_n(\mathbb{Q}_\ell) \end{array}$$

commute, where the top horizontal arrow is the natural ring homomorphism induced by ρ , and the vertical arrows are the natural inclusions.

Proof. Uniqueness is clear from the density of $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \otimes \mathbb{Q}_\ell$ in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$, so it suffices to construct $\tilde{\rho}$.

If $M \in \mathfrak{gl}_n(\mathbb{Q}_\ell)$, let

$$|M| := \ell^{\inf\{r \in \mathbb{Z} \mid \ell^r \cdot M \in \mathfrak{gl}_n(\mathbb{Z}_\ell)\}}.$$

Using the notation of Example 3.1.3, we choose topological generators $\lambda_1, \dots, \lambda_m$ of $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ and set $T_i = 1 + \lambda_i \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$. Then we set

$$\tilde{\rho} \left(\sum_I a_I T^I \right) = \sum_I a_I \rho(T^I).$$

This sum converges because $|\rho(T_i)| = |\rho(\gamma_i) - \text{Id}| \leq \ell^{-m}$, as ρ is trivial mod ℓ^m , so $|\rho(T^I)| \leq \ell^{-m|I|}$. But we have

$$|a_I \rho(T^I)| \leq |a_I| \ell^{-m|I|}$$

which tends to zero because $|a_I| < \ell^{r|I|}$ for $|I|$ large, with $r < m$.

To check continuity, we must check that $\tilde{\rho}$ is a bounded operator, i.e. that there exists $C > 0$ such that

$$\left| \tilde{\rho} \left(\sum_I a_I T^I \right) \right| \leq C \left| \sum_I a_I T^I \right|_r.$$

But we have

$$\left| \tilde{\rho} \left(\sum_I a_I T^I \right) \right| \leq \sup_I |a_I|_\ell \cdot |\tilde{\rho}(T^I)| \leq \sup_I \ell^{-v_\ell(a_I) - m|I|} \leq \sup_I \ell^{-v_\ell(a_I) - r|I|} = \left| \sum_I a_I T^I \right|_r,$$

so we may take $C = 1$. $\square$

Remark 3.1.5. The previous proposition is true in much greater generality (i.e. it holds if X satisfies Condition $\star$ below), but this case admits a simple proof and is all we require, as we may reduce our main theorems to the case where X is an affine curve.

3.2. Integral ℓ -adic periods. Theorems 2.3.1 and 2.2.4 imply that $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ admits a set of σ_α -eigenvectors with dense span (in the $\mathcal{S}$ -adic topology). The purpose of this section is to deduce an analogous statement for $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ for $r \gg 0$.

Consider the following condition on $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$:

$$\mathcal{S}^n / \mathcal{S}^{n+1} \text{ is } \mathbb{Z}_\ell\text{-torsion-free for all } n. \quad (\star)$$

This condition ensures that the natural map $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is injective, and hence that $\text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is $\mathbb{Z}_\ell$ -torsion-free for all i . Condition $\star$ is satisfied if e.g. X is a smooth curve; the case of an affine curve (which is all we require) follows immediately from the isomorphism of Example 2.1.2.

Theorem 3.2.1. *Let σ_α be as in Theorem 2.2.4, with α not a root of unity. Suppose X satisfies Condition $\star$. Then there exists $r_\alpha > 0$ such that for $r > r_\alpha$, $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ contains a set of σ_α -eigenvectors with dense span. Moreover $W^{-n} \subset \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of σ_α -eigenvectors with eigenvalues in $\{\alpha^n, \alpha^{n+1}, \dots\}$ and with dense span (in the topology defined by the r -Gauss norm).*

Equivalently, if y is a σ_α -eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$, $-v_n(\pi_n(y))$ grows at most linearly in n . We require several lemmas before giving the proof.

Lemma 3.2.2. *Consider $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ as a $\mathbb{Z}_\ell[\sigma_\alpha]$ -module, with the $W^\bullet$ -filtration. Suppose X satisfies Condition $\star$. Then for any $m \geq i + 1$,*

$$\text{Ext}_{\mathbb{Z}_\ell[\sigma_\alpha]}^1(W^{-i}/W^{-i-1}, W^{-i-1}/W^{-m})$$

is annihilated by $\ell^{v(i,m,\alpha)}$, where

$$v(i, m, \alpha) = \sum_{s=1}^{m-i-1} v_\ell(\alpha^s - 1).$$

Proof. Note that Condition $\star$ implies that the natural map $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is injective; hence W^{-s}/W^{-s+1} is $\mathbb{Z}_\ell$ -torsion-free for any s , and σ_α acts on it via the scalar α^s .

We proceed by induction on m ; the case $m = i + 1$ is trivial. For the induction step, from the short exact sequence

$$0 \rightarrow W^{-m+1}/W^{-m} \rightarrow W^{-i-1}/W^{-m} \rightarrow W^{-i-1}/W^{-m+1} \rightarrow 0$$

it suffices to show that $\text{Ext}_{\mathbb{Z}_\ell[\sigma_\alpha]}^1(W^{-i}/W^{-i-1}, W^{-m+1}/W^{-m})$ is annihilated by $\ell^{v_\ell(\alpha^{m-i-1}-1)}$. But σ_α acts via the scalar α^i on W^{-i}/W^{-i-1} and by α^{m-1} on W^{-m+1}/W^{-m} ; thus it is enough to show that

$$\text{Ext}_{\mathbb{Z}_\ell[\sigma_\alpha]}^1(\mathbb{Z}_\ell[\sigma_\alpha]/(\sigma_\alpha - \alpha^i), \mathbb{Z}_\ell[\sigma_\alpha]/(\sigma_\alpha - \alpha^{m-1}))$$

is annihilated by $\ell^{v_\ell(\alpha^{m-i-1}-1)}$. But this is clear from the free resolution

$$0 \longrightarrow \mathbb{Z}_\ell[\sigma_\alpha] \xrightarrow{\cdot(\sigma_\alpha - \alpha^i)} \mathbb{Z}_\ell[\sigma_\alpha] \longrightarrow \mathbb{Z}_\ell[\sigma_\alpha]/(\sigma_\alpha - \alpha^i) \longrightarrow 0.$$

□

Remark 3.2.3. Note that the proof of this lemma required the semi-simplicity of the σ_α -action on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ (i.e. the full strength of Theorem 2.2.4 — Lemma 2.2.6 does not suffice).

Remark 3.2.4. There are canonical classes $e_{i,m}(\sigma_\alpha, x) \in \text{Ext}_{\mathbb{Z}_\ell[\sigma_\alpha]}^1(W^{-i}/W^{-i-1}, W^{-i-1}/W^{-m})$ corresponding to the extensions

$$0 \rightarrow W^{-i-1}/W^{-m} \rightarrow W^{-i}/W^{-m} \rightarrow W^{-i}/W^{-i-1} \rightarrow 0.$$

We refer to the least b such that ℓ^b annihilates $e_{i,m}(\sigma_\alpha, x)$ as an *integral ℓ -adic period* of $\pi_1^\ell(X_{\bar{k}}, \bar{x})$, because these values measure the failure of the canonical σ_α -equivariant isomorphism

$$\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/W^{-m} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \xrightarrow{\sim} \bigoplus_{i=0}^{m-1} \text{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$$

to preserve the integral structures on both sides.

Lemma 3.2.5. *Let ℓ be a prime and $q \in (1 + \ell\mathbb{Z}_\ell)^\times$ if $\ell \neq 2$, $q \in (1 + 4\mathbb{Z}_2)^\times$ if $\ell = 2$. Then*

$$v_\ell(q^k - 1) = v_\ell(q - 1) + v_\ell(k).$$

Proof. This follows from the fact that $(1 + \ell\mathbb{Z}_\ell)^\times$ (resp. $(1 + 4\mathbb{Z}_2)^\times$) is pro-cyclic with $(1 + \ell\mathbb{Z}_\ell)^k = 1 + k\ell\mathbb{Z}_\ell$ (resp. $(1 + 4\mathbb{Z}_2)^k = 1 + 4k\mathbb{Z}_2$). □

Lemma 3.2.6. *Let ℓ be a prime and $q \in \mathbb{Z}_\ell^\times$. Let r be the order of q in $\mathbb{F}_\ell^*$ if $\ell \neq 2$ and in $(\mathbb{Z}/4\mathbb{Z}^*)$ if $\ell = 2$. Then*

$$v_\ell(q^k - 1) = \begin{cases} v_\ell(q^r - 1) + v_\ell(k/r) & \text{if } r \mid k \\ 0 & \text{if } r \nmid k \text{ and } \ell \neq 2 \\ 1 & \text{if } r \nmid k \text{ and } \ell = 2. \end{cases}$$

Proof. As in the statement, we consider the three cases $r \mid k$, $r \nmid k$ and $\ell \neq 2$, and $r \nmid k$ and $\ell = 2$. The first case is immediate from Lemma 3.2.5, replacing q with q^r and k with k/r . The other two cases follow from the definition of r . $\square$

Lemma 3.2.7. *Let q, ℓ, r be as in Corollary 3.2.6. Let*

$$C(q, \ell, k) = \frac{k}{r} \left(v_\ell(q^r - 1) + \frac{1}{\ell - 1} \right)$$

if $\ell \neq 2$ and

$$C(q, \ell, k) = \frac{k}{r} \left(v_\ell(q^r - 1) + \frac{1}{\ell - 1} + 1 \right) + \frac{1}{r}$$

if $\ell = 2$. Then

$$\sum_{i=1}^k v_\ell(q^i - 1) \leq C(q, \ell, k).$$

Proof. In the case $\ell \neq 2$, we have by Lemma 3.2.6 that $v_\ell(q^i - 1) = v_\ell(q^r - 1) + v_\ell(i/r)$ if $r \mid k$ and 0 otherwise. Thus we have

$$\sum_{i=0}^k v_\ell(q^i - 1) = \sum_{i=1}^{\lfloor k/r \rfloor} v_\ell(q^{ri} - 1) = \sum_{i=1}^{\lfloor k/r \rfloor} (v_\ell(q^r - 1) + v_\ell(i)).$$

But this last satisfies

$$\begin{aligned} \sum_{i=1}^{\lfloor k/r \rfloor} (v_\ell(q^r - 1) + v_\ell(i)) &= \lfloor k/r \rfloor v_\ell(q^r - 1) + \lfloor \frac{k}{r\ell} \rfloor + \lfloor \frac{k}{r\ell^2} \rfloor + \lfloor \frac{k}{r\ell^3} \rfloor + \cdots \\ &\leq \frac{k}{r} v_\ell(q^r - 1) + \frac{k}{r\ell(1 - \frac{1}{\ell})} \\ &= \frac{k}{r} \left(v_\ell(q^r - 1) + \frac{1}{\ell - 1} \right) \end{aligned}$$

In the case $\ell = 2$, and $r = 1$, an identical argument shows that

$$\sum_{i=0}^k v_\ell(q^i - 1) \leq \frac{k}{r} \left(v_\ell(q^r - 1) + \frac{1}{\ell - 1} \right).$$

If $\ell = 2$ and $r = 2$, we have

$$\begin{aligned} \sum_{i=1}^k v_\ell(q^i - 1) &= \sum_{i \text{ odd}, 1 \leq i \leq k} 1 + \sum_{i=1}^{\lfloor k/2 \rfloor} (v_\ell(q^r - 1) + v_\ell(i)) \\ &\leq \frac{k+1}{r} + \frac{k}{r} v_\ell(q^r - 1) + \frac{k}{\ell r} + \frac{k}{\ell^2 r} + \cdots \\ &= \frac{k}{r} \left(v_\ell(q^r - 1) + \frac{1}{\ell - 1} + 1 \right) + \frac{1}{r}, \end{aligned}$$

which concludes the proof. $\square$

Proof of Theorem 3.2.1. We claim that we may take $r_\alpha = 2C(\alpha, \ell, 1)$, defined as in Lemma 3.2.7. We first show that for $r > r_\alpha$, every σ_α -eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ in fact lies in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$.

We claim that there exist unique σ_α -equivariant splittings s_i of the quotient maps

$$W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} \rightarrow \mathrm{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}},$$

for any $r > r_\alpha$; this will imply that any σ_α -eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ in fact lies in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$. Indeed, by Lemma 3.2.2 and the Yoneda interpretation of Ext^1 in terms of extension classes, there exists a σ_α -equivariant map

$$s_i^m : \text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] / W^{-m} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$$

such that the diagram

$$\begin{array}{ccccccc} & & & \text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] & & & \\ & & \swarrow s_i^m & \downarrow \ell^{v(i, m, \alpha)} & & & \\ 0 & \longrightarrow & W^{-i-1}/W^{-m} & \longrightarrow & W^{-i}/W^{-m} & \longrightarrow & \text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \longrightarrow 0 \end{array}$$

commutes, where W^{-i} above denotes $W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$, and $v(i, m, \alpha)$ is defined as in Lemma 3.2.2. Moreover s_i^m is unique because α does not appear as an eigenvalue for the action of σ_α on W^{-i-1}/W^{-m} . By uniqueness, the diagram

$$\begin{array}{ccc} & & \text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \\ & \swarrow \ell^{-v(i, m+1, \alpha)} \cdot s_i^{m+1} & \downarrow \ell^{-v(i, m, \alpha)} \cdot s_i^m \\ W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] / W^{-m-1} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] & \longrightarrow & W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] / W^{-m} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \end{array}$$

commutes, so, extending scalars, the maps $\{\ell^{-v(i, m, \alpha)} \cdot s_i^m\}_{m > i}$ define (by completeness of $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ with respect to the $\mathcal{I}$ -adic, hence $W^\bullet$ filtration) a σ_α -equivariant map

$$\tilde{s}_i : \text{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]],$$

splitting the natural quotient map $W^{-i} \rightarrow \text{gr}_W^{-i}$. We claim this map factors through $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$ for $r > 2C(\alpha, \ell, 1)$, giving the desired maps s_i .

But indeed, for $y \in \text{gr}_W^{-i} \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$,

$$v_n(\pi_n(\tilde{s}_i(y))) \geq -v(i, 2n, \alpha)$$

by definition of the $\tilde{s}_i$ and by Proposition 2.2.3 (using that $W^{-2n-1} \subset \mathcal{I}^n$). But by Lemma 3.2.7,

$$v(i, 2n, \alpha) \leq C(\alpha, \ell, 2n - i - 1);$$

and $C(\alpha, \ell, 2n - i - 1) - nr \rightarrow \infty$ as $n \rightarrow \infty$, as we've chosen $r > 2C(\alpha, \ell, 1)$. Thus

$$v_n(\pi_n(\tilde{s}_i(y))) + nr \rightarrow \infty$$

and $\tilde{s}_i$ factors through $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$ as desired. Let

$$s_i : \text{gr}_W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \rightarrow W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$$

be the induced map.

Now let $y \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ be a non-zero σ_α -eigenvector; then there exists some maximal i such that $y \in W^{-i}$, as the W -adic topology is separated by Proposition 2.2.3. Then we have $\sigma_\alpha y = \alpha^i y$, because $y \neq 0 \bmod W^{-i-1}$. Now we claim $y = s_i(y \bmod W^{-i-1})$; indeed, $y - s_i(y \bmod W^{-i-1})$ is a σ_α -eigenvector with eigenvalue α^i , contained in W^{-i-1} , hence zero. Thus y is in $W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$, as desired.

We now check that the σ_α -eigenvectors in $W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ are dense in the topology defined by the Gauss norm. Given $z \in W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$, let $z_0 = z, w_0 = s_i(z_0 \bmod W^{-i-1})$ and in general,

$$z_j = z_{j-1} - w_{j-1}, w_j = s_{i+j}(z_j \bmod W^{-i-j-1}).$$

Then the w_j are σ_α -eigenvectors, so it suffices to show that

$$\sum w_j \rightarrow z,$$

or equivalently that $|z_n|_r \rightarrow 0$. We leave this as an elementary exercise, again using Lemma 3.2.7. $\square$

Remark 3.2.8. Note that we can take $r_\alpha = 2C(\alpha, \ell, 1)$, defined as in Lemma 3.2.7, by the very first line in the proof above. If $H^1(X_{\bar{k}}, \mathbb{Q}_\ell)$ is pure of weight i ($i = 1, 2$), we may take $r_\alpha = C(\alpha^i, \ell, 1)$ (as in this case the weight filtration equals the $\mathcal{I}$ -adic filtration, up to renumbering).

4. APPLICATIONS TO ARITHMETIC AND GEOMETRIC REPRESENTATIONS

4.1. Main Results. We now prepare for the proof of Theorem 1.1.2.

Lemma 4.1.1. *Suppose that*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$$

is an arithmetic representation (defined as in Definition 1.1.1), which is trivial mod ℓ^r , and that $\rho \otimes \mathbb{Q}_\ell$ is irreducible. Then there is a finite extension $k \subset k'$ and a representation $\beta : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$ such that

- ρ is a subquotient of $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$, and
- $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ is trivial mod ℓ^r .

Proof. By assumption, there exists a finite extension $k \subset k'$ and a representation

$$\gamma : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_\ell)$$

such that ρ arises as a subquotient of $\gamma|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$. Let $0 = V^0 \subset V^1 \subset \dots \subset V^r = \gamma$ be the socle filtration of $\gamma|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})} \otimes \mathbb{Q}_\ell$ (viewed as a representation of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$), i.e. V^i is the largest subrepresentation of $\gamma|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ such that V^i/V^{i-1} is semi-simple as a $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ -representation. As the socle filtration is canonical, each V^i/V^{i-1} also extends to a representation of $\pi_1^{\text{ét}}(X_{k'}, \bar{k})$. As $\rho \otimes \mathbb{Q}_\ell$ is irreducible, it arises as a direct summand of V^i/V^{i-1} for some i (viewed as a $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ -representation). Thus by replacing γ with $(V^i \cap \gamma)/(V^{i-1} \cap \gamma)$, we may assume $\gamma|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})} \otimes \mathbb{Q}_\ell$ is semisimple.

Let $W \subset V^i/V^{i-1}$ be the minimal sub- $\pi_1^{\text{ét}}(X_{k'}, \bar{x})$ -representation containing $\rho \otimes \mathbb{Q}_\ell$. As a $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ -representation, W splits as a finite direct sum of the form

$$W = \bigoplus (\rho^\sigma \otimes \mathbb{Q}_\ell)^{\alpha_\sigma},$$

where ρ^σ denotes the representation obtained by pre-composing ρ with an automorphism σ of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$. Let β be the $\mathbb{Z}_\ell$ -submodule of W spanned by ρ^τ , for $\tau \in \pi_1^{\text{ét}}(X_{k'}, \bar{x})$. β is a lattice by the compactness of G_k , and ρ is clearly a subquotient of β . Moreover β is spanned by the modules ρ^τ , each of which is trivial mod ℓ^r (as representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$), and hence is trivial mod ℓ^r as desired. $\square$

Proof of Theorem 1.1.2. By [Del81], there exists a curve C/k and a map $C \rightarrow X$ such that

$$\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}) \rightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$$

is surjective for any geometric point $\bar{x}$ of C . We may assume C is affine by deleting a point, which does not affect the surjectivity of the map in question. Thus we may immediately replace X with C ; observe that $\pi_1^{\ell}(C_{\bar{k}}, \bar{x})$ is a finitely-generated free pro- ℓ group and in particular satisfies Condition $\star$. After replacing k with a finite extension k' , we may assume $\bar{x}$ comes from a rational point of C , giving a natural action of $G_{k'}$ on $\pi_1^{\ell}(C_{\bar{k}}, \bar{x})$. Thus we replace k with k' and X with C (and we rename C as X in what follows).

By Theorem 2.2.4, there exists $\alpha \in \mathbb{Z}_{\ell}^{\times}$, not a root of unity, and $\sigma_{\alpha} \in G_k$ such that σ_{α} acts on $\text{gr}_W^{-i} \mathbb{Q}_{\ell}[[\pi_1^{\text{ét}, \ell}(X_{\bar{k}}, \bar{x})]]$ via $\alpha^i \cdot \text{Id}$. Then by Theorem 3.2.1, there exists $r_{\alpha} > 0$ such that the σ_{α} -action on $\mathbb{Q}_{\ell}[[\pi_1^{\ell}(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of eigenvectors with span dense in the r -Gauss norm topology, for any $r > r_{\alpha}$. Set N to be the least integer greater than r_{α} , and choose r with $r_{\alpha} < r < N$.

Now let

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_{\ell})$$

be an arithmetic representation, as in the statement of the theorem, such that ρ is trivial mod ℓ^N . We claim that ρ is in fact unipotent. Let $0 = V_0 \subset V_1 \subset \dots$ be the socle filtration of $\rho \otimes \mathbb{Q}_{\ell}$; then replacing ρ with $(\rho \cap V_i)/(\rho \cap V_{i-1})$ (which is arithmetic, as it is a subquotient of an arithmetic representation), we may assume $\rho \otimes \mathbb{Q}_{\ell}$ is semisimple, say $\rho \otimes \mathbb{Q}_{\ell} = \oplus_i W_i$ with W_i irreducible. Replacing ρ with $\rho \cap W_i$ for any i , we may assume $\rho \otimes \mathbb{Q}_{\ell}$ is irreducible. Now we may apply Lemma 4.1.1, so we may (at the cost of losing irreducibility), assume that there exists a finite extension k'/k and a continuous representation

$$\beta : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \rightarrow GL_n(\mathbb{Z}_{\ell})$$

such that $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ is trivial mod ℓ^N , and such that ρ is a subquotient of $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$. It suffices to show that $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ is unipotent.

Let m be such that $\sigma_{\alpha}^m \in G_{k'}$.

As

$$\ker(GL_n(\mathbb{Z}_{\ell}) \rightarrow GL_n(\mathbb{Z}/\ell^N \mathbb{Z}))$$

is a pro- ℓ -group, $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ factors through $\pi_1^{\ell}(X_{\bar{k}}, \bar{x})$. Thus by Proposition 3.1.4, β induces a σ_{α}^m -equivariant map

$$\mathbb{Q}_{\ell}[[\pi_1^{\ell}(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}} \xrightarrow{\tilde{\beta}} \mathfrak{gl}_n(\mathbb{Q}_{\ell}),$$

where the σ_{α}^m -action on $\pi_1^{\ell}(X_{\bar{k}}, \bar{x})$ comes from our choice of rational basepoint, and the action on $\mathfrak{gl}_n(\mathbb{Q}_{\ell})$ comes from the fact that $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$ extends to a representation of $\pi_1^{\text{ét}}(X_{k'}, \bar{x})$, by the previous paragraph.

Now $\mathfrak{gl}_n(\mathbb{Q}_{\ell})$ is a finite-dimensional vector space, so the action of σ_{α}^m on $\mathfrak{gl}_n(\mathbb{Q}_{\ell})$ only has finitely many eigenvalues. In particular, for $s \gg 0$, α^s is not an eigenvalue of the σ_{α}^m -action on $\mathfrak{gl}_n(\mathbb{Q}_{\ell})$. Thus by Theorem 3.2.1,

$$\tilde{\beta}(W^{-i} \mathbb{Q}_{\ell}[[\pi_1^{\ell}(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}) = 0$$

for $i \gg 0$, as $W^{-i} \mathbb{Q}_{\ell}[[\pi_1^{\ell}(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of σ_{α} -eigenvectors with dense span and with associated eigenvalues in $\{\alpha^i, \alpha^{i+1}, \dots\}$. Hence in particular $\tilde{\beta}(W^{-i} \mathbb{Z}_{\ell}[[\pi_1^{\ell}(X_{\bar{k}}, \bar{x})]]) =$

0, where we may consider $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ as a sub-algebra of $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ because Condition $\star$ is satisfied by the first paragraph of this proof.

But by Proposition 2.2.3 (and again using Condition $\star$), the W -adic topology on $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is the same as the $\mathcal{I}$ -adic topology — hence $\tilde{\beta}(\mathcal{I}^t) = 0$ for $t \gg 0$. But this immediately implies that $\beta|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$, and hence ρ , is unipotent as desired. $\square$

Remark 4.1.2. Observe that N only depends on X and ℓ , and not on any of the parameters of ρ (for example, its dimension). To our knowledge this was not expected. Indeed, if X is an affine curve, $N(X, \ell)$ only depends on the index of the image of the natural Galois representation

$$G_k \rightarrow GL(H^1(X_{\bar{k}}, \mathbb{Z}_\ell))$$

in its Zariski-closure, by the proof of Theorem 2.2.4. If this index is 1 for some $\ell > 2$ (as is expected to be the case for almost all ℓ (see e.g. [Ser94, §10], [Win02], and [CM15] for discussion of the case where $H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$ is pure of weight 1 — as far as we know, the mixed case has not been conjectured in the literature, though it seems natural to do so), we may take $N(X, \ell) = 1$, by choosing α in Theorem 2.2.4 to be a topological generator of $\mathbb{Z}_\ell^\times$, and using Remark 3.2.8.

We may now prove Corollary 1.1.4.

Proof of Corollary 1.1.4. Let $k_0 \subset k$ be a finitely generated subfield over which X is defined; suppose that ρ arises as a subquotient of the monodromy representation on $R^i \pi_* \mathbb{Z}_\ell$ for some smooth proper morphism $\pi : Y \rightarrow X$. Then there exists a finitely generated k_0 -algebra $R \subset k$, a proper R -scheme $\mathcal{Y}$, and a smooth proper map of R -schemes

$$\tilde{\pi} : \mathcal{Y} \rightarrow X_R$$

such that π is the base change of $\tilde{\pi}$ along the inclusion $R \subset k$. Quotienting out torsion we may replace $R^i \tilde{\pi}_* \mathbb{Z}_\ell$ by a lisse subquotient. Now specializing to any closed point of $\text{Spec}(R)$, we see that $\rho|_{\pi_1(X_{\bar{k}})}$ is arithmetic, so we are done by Theorem 1.1.2.

Observe that the N from Theorem 1.1.2 only depends on our choice of model of X over k_0 , which do not depend on ρ in any way, so we have the desired uniformity in ρ . $\square$

4.2. Explicit Bounds. In some cases, one can make the N appearing in Theorem 1.1.2 explicit. Indeed, if C is an affine curve $N(C, \ell)$ only depends on the index of the representation

$$G_k \rightarrow GL(H^1(C_{\bar{k}}, \mathbb{Z}_\ell))$$

in its Zariski closure (see Remark 4.1.2). Thus we now turn to cases where we understand this representation.

Our main example will be the case

$$X = \mathbb{P}_k^1 \setminus \{x_1, \dots, x_n\},$$

where $x_1, \dots, x_n \in \mathbb{P}^1(k)$. Unlike Theorem 1.1.2, this theorem works in arbitrary characteristic.

Theorem 4.2.1. *Let k be a finitely generated field, and let*

$$X = \mathbb{P}_k^1 \setminus \{x_1, \dots, x_n\},$$

for $x_1, \dots, x_n \in \mathbb{P}^1(k)$. Let ℓ be prime different from the characteristic of k and $q \in \mathbb{Z}_\ell^\times$ any element of the image of the cyclotomic character

$$\chi : \text{Gal}(\bar{k}/k) \rightarrow \mathbb{Z}_\ell^\times;$$

let s be the order of q in $\mathbb{F}_\ell^\times$ if $\ell \neq 2$ and in $(\mathbb{Z}/4\mathbb{Z})^\times$ if $\ell = 2$. Let $\epsilon = 1$ if $\ell = 2$ and 0 otherwise. Let

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \rightarrow GL_m(\mathbb{Z}_\ell)$$

be an arithmetic representation such that $\rho|_{\pi_1(X_{\bar{k}}, \bar{x})}$ is trivial mod ℓ^N , with

$$N > \frac{1}{s} \left(v_\ell(q^s - 1) + \frac{1}{\ell - 1} + \epsilon \right).$$

Then ρ is unipotent.

Remark 4.2.2. For any X , the bound above becomes

$$N > \frac{1}{\ell - 1} + \frac{1}{(\ell - 1)^2}$$

for odd ℓ , if q is a topological generator of $\mathbb{Z}_\ell^\times$. And we may choose q to be a topological generator if $\ell > 2$ and the cyclotomic character

$$\chi : \text{Gal}(\bar{k}/k) \rightarrow \mathbb{Z}_\ell^\times$$

is surjective. Thus in particular if χ is surjective at ℓ for some odd ℓ (note that this holds for almost all ℓ), then any arithmetic representation of $\pi_1^{\text{ét}}(X_{\bar{k}})$, which is trivial mod ℓ , is unipotent.

Proof of Theorem 4.2.1. We first choose a rational (or rational tangential) basepoint x of X over k ; we may always choose a rational tangential basepoint as the $x_i \in \mathbb{P}^1(k)$. Let $\sigma_q \in G_k$ be such that $\chi(\sigma_q) = q$. Choose r with

$$N > r > \frac{1}{s} \left(v_\ell(q^s - 1) + \frac{1}{\ell - 1} + \epsilon \right) = C(q, \ell, 1),$$

where $C(q, \ell, 1)$ is defined as in Lemma 3.2.7.

In this case,

$$H^1(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(1)^{n-1},$$

so σ_q satisfies the conclusions of Theorem 2.2.4. Thus by Theorem 3.2.1 and Remark 3.2.8 (using that $r > C(q, \ell, 1)$), $W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell-r}$ admits a set of σ_q -eigenvectors with dense span in the Gauss norm topology, and with eigenvalues in $\{q^i, q^{i+1}, \dots\}$. Now we conclude by precisely imitating the proof of Theorem 1.1.2. $\square$

Corollary 4.2.3. *Let k be a field with prime subfield k_0 , and let*

$$X = \mathbb{P}_{\bar{k}}^1 \setminus \{x_1, \dots, x_n\}.$$

Let K be the field generated by cross-ratios of the x_i , that is,

$$K = k_0 \left(\frac{x_a - x_b}{x_c - x_d} \right)_{1 \leq a < b < c < d \leq n}.$$

Let ℓ be a prime different from the characteristic of k and $q \in \mathbb{Z}_\ell^\times$ be any element of the image of the cyclotomic character

$$\chi : \text{Gal}(\bar{K}/K) \rightarrow \mathbb{Z}_\ell^\times;$$

let s be the order of q in $\mathbb{F}_\ell^\times$ if $\ell \neq 2$ and in $(\mathbb{Z}/4\mathbb{Z})^\times$ if $\ell = 2$. Let $\epsilon = 1$ if $\ell = 2$ and 0 otherwise. Let

$$\rho : \pi_1^{\text{ét}}(X, x) \rightarrow GL_m(\mathbb{Z}_\ell)$$

be a continuous representation which is trivial mod ℓ^N , with

$$N > \frac{1}{s} \left(v_\ell(q^s - 1) + \frac{1}{\ell - 1} + \epsilon \right).$$

If ρ is geometric, then it is unipotent.

Proof. We let $X = \mathbb{P}_k^1 \setminus \{x_1, \dots, x_n\}$. Then X admits a natural model over the field

$$K = k_0 \left(\frac{x_a - x_b}{x_c - x_d} \right)_{1 \leq a < b < c < d \leq n}.$$

Now the result follows from Theorem 4.2.1 in a manner identical to the deduction of Corollary 1.1.4 from Theorem 1.1.2. $\square$

Remark 4.2.4. If $k = \mathbb{Q}$ in Theorem 4.2.1, or if $K = \mathbb{Q}$ in Corollary 4.2.3, we may take $N(X, \ell) = 1$ for any odd ℓ , by Remark 4.2.2.

We give one final corollary over fields of arbitrary characteristic. Observe that if there is a map $\mathbb{P}^1 \setminus \{x_1, \dots, x_n\} \rightarrow X$ which induces a surjection on geometric pro- ℓ fundamental groups, we may apply Theorem 4.2.3 to find restrictions on arithmetic representations of the fundamental group of X . Such a map exists if X is an open subvariety of a separably rationally connected variety, by the main result of [Kol03]. Thus we have

Corollary 4.2.5. *Let k be a finitely generated field and X an open subset of a separably rationally connected k -variety. Let ℓ be a prime different from the characteristic of k . Then there exists $N = N(X, \ell)$ such that if*

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \rightarrow GL_m(\mathbb{Z}_\ell)$$

is arithmetic, and is trivial mod ℓ^N , then ρ is unipotent.

4.3. Sharpness of results and examples. The hypothesis on the cyclotomic character in Theorem 4.2.1 and Corollary 4.2.3 may seem strange, but they are in fact necessary.

Example 4.3.1. Let $\ell = 3$ or 5, and consider the (connected) modular curve $Y(\ell)$, parametrizing elliptic curves with full level ℓ structure. $Y(\ell)$ has genus zero. Let $E \rightarrow Y(\ell)$ be the universal family, and $\bar{x}$ any geometric point of $Y(\ell)$. Then the tautological representation

$$\rho : \pi_1^{\text{ét}}(Y(\ell), \bar{x}) \rightarrow GL(T_\ell(E_{\bar{x}}))$$

is trivial mod ℓ , and $\rho|_{\pi_1^{\text{ét}}(Y(\ell)_{\bar{k}}, \bar{x})}$ is evidently both arithmetic and geometric. This does not contradict Theorem 4.2.1, Corollary 4.2.3, or Remark 4.2.4 because the field generated by the cross-ratios of the cusps of $Y(\ell)$ is $\mathbb{Q}(\zeta_\ell)$, whose cyclotomic character is not surjective at ℓ .

We may use these results to construct example of representations of fundamental groups which do not come from geometry — indeed, the following is an example of a representation which is not arithmetic or geometric, and which we do not know how to rule out by other means:

Example 4.3.2. As before, let $Y(3)$ be the modular curve parametrizing elliptic curves with full level three structure. Then

$$Y(3)_{\mathbb{C}} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}$$

where $\lambda \in \mathbb{Q}(\zeta_3)$; let $x \in Y(3)(\mathbb{C})$ be a point. Let ρ be the tautological representation

$$\rho : \pi_1(Y(3)(\mathbb{C})^{\text{an}}, x) \rightarrow GL(H^1(E_x(\mathbb{C})^{\text{an}}, \mathbb{Z})).$$

Let

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty, \beta\}$$

where $\beta \in \mathbb{Q} \setminus \{0, 1\}$. Then $X(\mathbb{C})^{\text{an}}$ is homeomorphic to $Y(3)(\mathbb{C})^{\text{an}}$ (indeed, both are homeomorphic to a four-times-punctured sphere); let

$$j : X(\mathbb{C})^{\text{an}} \rightarrow Y(3)(\mathbb{C})^{\text{an}}$$

be such a homeomorphism. Then the representation

$$\tilde{\rho} : \pi_1(X(\mathbb{C})^{\text{an}}, j^{-1}(x)) \xrightarrow{j^*} \pi_1(Y(3)(\mathbb{C})^{\text{an}}, x) \xrightarrow{\rho} GL(H^1(E_x(\mathbb{C})^{\text{an}}, \mathbb{Z}))$$

is trivial mod 3 and thus cannot arise from geometry, by Remark 4.2.4. Likewise, the representation

$$\pi_1^{\text{ét}}(X) \rightarrow GL(H^1(E_x, \mathbb{Z}_\ell))$$

obtained from $\tilde{\rho}$ cannot be arithmetic, again by Remark 4.2.4.

We do not know a way to see this using pre-existing methods, since any criterion ruling out this representation would have to detect the difference between X and $Y(3)$; for example, the quasi-unipotent local monodromy theorem does not rule out $\tilde{\rho}$. This example was suggested to the author by George Boxer.

Example 4.3.3. Let $X/\mathbb{Q}$ be a proper genus two curve; recall that

$$\pi_1(X(\mathbb{C})^{\text{an}}) = \langle a_1, b_1, a_2, b_2 \rangle / ([a_1, b_1][a_2, b_2] = 1).$$

Let ℓ be a prime and A, B non-unipotent $n \times n$ integer matrices which are equal to the identity mod ℓ^N . Then for $N \gg 0$, Corollary 1.1.4 implies that the representation

$$a_1 \mapsto A, b_1 \mapsto B, a_2 \mapsto B, b_2 \mapsto A$$

does not come from geometry; likewise the induced representation

$$\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) \rightarrow GL_n(\mathbb{Z}_\ell)$$

is not arithmetic by Theorem 1.1.2. Again, we do not know how to see this using previously known results.

On the other hand, we do not expect our bounds on r_α in Theorem 3.2.1 to be sharp for arbitrary X . In some cases, however, one may bound r_α from below by finding a non-unipotent arithmetic representation of $\pi_1(X_{\bar{k}})$ trivial mod ℓ^M . In this case, by the contrapositive of Theorem 1.1.2, we have $r_\alpha \geq M$ for any σ_α ; for example, by considering the tautological monodromy representation of the fundamental group of the modular curve $Y(\ell^M)$, we find that $r_\alpha \geq M$ for any σ_α acting on the fundamental group of $Y(\ell^M)$.

We were also (with the exception of Theorem 4.2.1 and Corollary 4.2.3) unable to prove results in positive characteristic, because ℓ -adic open image theorems as in [Bog80] are no longer true in this case, so the proof of Lemma 2.2.6 fails. We could salvage this situation if the following question has a positive answer:

Question 4.3.4. Let X be a smooth curve over a finite field k , and let ℓ be a prime different from the characteristic of k . Does there exist an $r = r(X)$ such that $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of Frobenius eigenvectors with dense span?

As in Theorem 3.2.1, one would need to show that for a Frobenius eigenvector $y \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$, one has $|v_n(\pi_n(y))| = O(n)$. We can show, using Yu's p -adic Baker's theorem on linear forms in logarithms [Yu07], that for almost all ℓ , $|v_n(\pi_n(y))| = O(n \log n)$, which does not suffice for our applications.

One could also ask for stronger uniformity in geometric invariants of X . For example, a positive answer to the following question would imply positive answer to a “pro- ℓ ” version of the geometric torsion conjecture, by the methods of this section.

Question 4.3.5. Let X be a smooth curve over a finite field k , and let ℓ be a prime different from the characteristic of k . Does there exist an $r = r(\text{gonality}(X), \ell) > 0$, depending only on the gonality of X and on ℓ , such that $\overline{\mathbb{Q}_\ell}[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\leq \ell^{-r}}$ admits a set of Frobenius eigenvectors with dense span? If so, does r tend to zero rapidly with ℓ ?

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