Dan and Arnav: The "nonabelian Poincaré duality" circle of ideas.

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1 Dan Litt: November XKCD Seminar: Duality and Generalized Scanning

I want to begin by motivating this talk with some theorems which are all about configuration spaces and labeled configuration spaces, and talk about the relation to duality. Probably some of this is well-known.

1.1

The story about scanning started with the following theorem of MacDuff. That's Dusa MacDuff, not the character McDuff from Shakespeare's play Macbeth.

Theorem 1. *X a pointed space, M an n-manifold, Then*

$$Conf_X(M) \longrightarrow \Gamma_c(T_M^\infty \wedge_M X)$$

is a weak equivalence if M is compact without boundary OR X is connected.

Here the configuration space $Conf_X(M)$ is

$$\left(\coprod Conf_i(M) \times_{\Sigma_i} X^i\right) / \sim$$

where the relation is $(m, x) \sim (m - m_i, x - x_i)$ if $m_i \in \partial M$, $x_i = x$. [FIX THIS]
And... this is something Arnav likes a lot, as do I.

Theorem 2. *(Dold-Thom) Let $Sym^\infty(X, A)$ be the space of unordered points in X, labeled by A an abelian group. Then*

$$\pi_i\left(Sym^\infty(X, A)\right) = H_i(X, A)$$

if X is a connected CW-complex.

And there are a lot of generalizations, due to Salvatore and other people....
Example.

$$Sym^n S^2 = Sym^n \mathbb{CP}^2 = \mathbb{CP}^n$$

And in particular we see that

$$\pi_i(\mathbb{CP}^\infty) = H_i(S^2, \mathbb{Z})$$

So this is a way of making Eilenberg-MacLane spaces out of Moore spaces.
Example. IF M is parallelizable, X is connected, then

$$Conf_X(M) \xrightarrow{\sim} Map_c(M, \Sigma^n X)$$

Example.

$$Conf_X(\mathbb{R}) = \Omega\Sigma X.$$

1.2

Now I want to talk about two fundamental examples. The goal of this talk will be to go from these two examples to the whole theorem.

Example.

$$Sym^\infty(D^n, A) = A$$

Example.

$$Conf_X(\overline{D^m}) = \Sigma^n X$$

Proof. I'm going to let $Conf_X(\overline{D_n})_{\leq n}$ be configurations with at most n points, and I claim that

$$Conf_X(\overline{D_n})_{\leq 1} = \left(\overline{D^n} \times X\right)/S^{n-1} \times X = \Sigma^n X$$

You scale stuff towards the boundary... and when it hits the boundary you can move down to level $n-1$.

Let's write down the annoying diagram.

$$\begin{tikzcd}
Conf_X(\overline{D^n}) \times [0,1] \arrow[r, "scaling"] & Conf_X(\overline{D^n}) \\
Conf_X(\overline{D^n}) \arrow[u] \arrow[r, "centralpoint"] & Conf_X(\overline{D^n})_{\leq 1} \arrow[u]
\end{tikzcd}$$

Pardon: The bottom map certainly doesn't seem to be continuous.

Dan: Hmm...

Pardon: Well, there is some diagram that works.

Dan: I don't really care about this, and I think you can tell.

1.3

Unfortunately, theorems are not a sheaf, so you can't prove theorems locally. But I'll try to make some kind of sheafy argument, gluing together the claims I made (above incorrectly).

Theorem 3. *(Refined Dold-Thom) Suppose X is a compact oriented manifold.*

$$X \xrightarrow{[\Delta]} Maps(X, B^{dim X}\mathbb{Z})$$

The righthand side is an abelian group, so we get maps

$$Sym^\infty(X) \longrightarrow Maps(X, B^n\mathbb{Z}),$$

and the statement is that this is a weak equivalence if X is connected.

So now how do we make this work if X is not a manifold? (We used Poincaré duality in getting a cohomology class $\alpha \in H^*(X \times X; \mathbb{Z})$, the (dual of the) fundamental class of the diagonal map.)

I will spend the rest of the time explaining how. We'll construct a map

$$X \longrightarrow \Gamma_c(X, DK(RHom(\omega_X, \underline{\mathbb{Z}})))$$

and hopefully this is something we can construct in some generality.

Here "DK" stands for Dold-Kan, and....

Well, I have to tell you what the rest of these things mean.

X will be... well, a connected CW complex will be fine.

1.4

Okay, we need an interlude on Verdier duality.

Uh, so what's the deal with sheaves?

Suppose

$$f : X \longrightarrow Y$$

is a map of spaces, and then there's an adjoint pair

$$f^* : Shv(Y) \leftrightarrows Shv(X) : f_*$$

Here, f^* doesn't change in the derived setting.

$$f^* : D^b(Y) \leftrightarrows D^b(X) : Rf_*$$

If we don't have a finiteness condition, we only have Poincaré duality for cohomology with compact support, right? So we're looking for something like that. Here Verdier duality says

$$Rf_! : D^b(X) \leftrightarrows D^b(Y) : Rf^!$$

Here $Rf_!$ means cohomology with compact support along fibers, and $Rf^!$ is some mysterious adjoint, but which we can construct with homotopy theory, using Brown representability.

So, Example. If $f : M \to pt$, where M is a manifold, then $Rf^!k$ is the orientation sheaf.

Remark: By formal properties of adjointness, there is a trace map

$$R\pi_! R\pi^! k \longrightarrow k$$

Remark: There is some analogous upper shrief for spectra, by Brown representability, which I don't know anything about, but it would be interesting to see whether one can duplicate this whole story for configuration spaces of spectra (sheaves of spectra, i.e. parametrized spectra).

1.5

Okay. So let me give a brief interlude on the Dold-Kan construction. So the original theorem of Dold and Kan is that

Theorem 4. *There is an equivalence of categories*

$$DK : Ch^*(A) \longrightarrow cAb : N$$

Here

$$N(X_\bullet)_n = \bigcap_{i>0} Kerd_i$$

and

$$DK(A_\bullet)[n] = Hom(N(\mathbb{Z}\Delta[n]), A_\bullet)$$

[Cary asked about getting a Quillen equivalence of model categories, but Dan said he doesn't want to talk about model structures.]

Two more useful things: 1) This gives an equivalence between preshaves of chain complexes of abelian groups and simplicial presheaves of abelian groups. 2) To a presheaf of chain complexes of abelian groups, we can associate a space computing the cohomology of the sheafification.

What this means is... given $F^\bullet$, we get something such that ??? $\pi_*(Sections)$ is $H^*(F^\bullet)$ and $\pi_i(\Gamma_c)$ is $H^i_c(F^\bullet)$.

[Finally I was unable to keep up with the rapid stream of material.]

1.6

I'm not going to define the scanning map; you should look that up on your own. But I do have some questions.

1. The ingredients of this proof exist for spectra. So there should be a proof of Dold-Thom for configurations in a parametrized spectrum. [Cary: That sounds like Poincaré Duality for spectra.]

2. If you only have partially summable lables, how do you make $H(F)$? [Cary: So like, redoing Salvatore's stuff?] There should be a "dualizing space", but I don't know what that means.

3. [cosheaves on open sets vs. sheaves on closed sets as in e.g. Pardon]

2 Arnav: Dec. 10. Étale-cohomological Dold-Thom

2.1 Symmetric Powers: What's the deal?

Okay, X a connected topological space. And, eh, let's make it pointed.

$$Sym^n X = X^n/\Sigma_n$$

and this is a naive quotient, not something more complicated like a homotopy quotient. This is the simplest example of a configuration space, which are all the rage these days.

The simplest example of a theorem about these would be Dold-Thom. I can take the colimit of these maps

$$Sym^n X \longrightarrow Sym^{n+1} X$$

and get something called $Sym^\infty X$. And Dold-Thom says that

Theorem 5. *(Dold-Thom)*

$$\pi_* Sym^\infty X \cong \tilde{H}_*(X;\mathbb{Z})$$

And since almost everyone here is a topologist, and I mean I think everyone is except me,

$$Sym^\infty X \cong (X \wedge H\mathbb{Z})_0.$$

[Some quibbling about this notation.]

One thing I'm interested in is stability of these configuration spaces which is also all the range.

Theorem 6.

$$\pi_k Sym^n X \xrightarrow{\sim} \pi_k Sym^{n+1} X$$

is an isomorphism for $k \leq n$.

Because as n increases, you're, like, only adding higher cells. (That's almost entirely a proof. But, the only issue is, not only do I get products of cells, I'm getting symmetric powers of cells. The issue is that...)

Great, this is like the simplest example of stability for these guys.

Sander: It's not $k \leq n$ for $n = 1$.

In fact, the content of the statement for $k = 1$ is that Sym^∞ just abelianizes, since reduced H_1 is the abelianization of π_1 for X pointed connected.

Yeah, so, back to the issue with that naive idea for a proof that I mentioned. The proof works if I can prove the following connectivity estimate:

$$S^{nk-1}/\Sigma^n$$

is $(n-1)$-connected. This is a sphere inside k copies of standard rep. of Σ_n.

I was unable to prove this statement just by mucking around. Eventually Sander was like, did you try reading anything? And I was like, no. And he told me how to do it using standard techniques. You use the techniques that people like O. Randal-Williams and Soren Galatius have come up with for dealing with homological stability for configuration spaces.

Aaron: Does it have something to do with like equivariant cell decompositions?

That's what I thought, but then I realized I don't know equivariant homotopy theory either.

I feel like there may be a nice proof along these lines, but I don't know.

Aaron: That may sit in a cofiber sequence, and then you could try to use equivariant Hurewicz. But then weird things come up, like, what's equivariant connectivity as a function, rather than a number...

2.2

Surprise! I'm an algebraic geometer.

We have configuration spaces in algebraic geometry, too. Or we might like to call them moduli spaces.

We still have

$$Sym^n X = X^n/\Sigma_n$$

but this isn't some naive orbit space now, we have to endow it with the structure of a scheme.

Now we're considering some map of schemes

$$SpecA \longrightarrow Spec(A^{\otimes n})^{\Sigma_n}$$

Sander: Is it easy to say what it is using like functor of points formalism?

I don't think so. It isn't like a free quotient or anything.

Someone: It seems like you can try to define it using functor of points but then you have to sheafify over the étale sheaf or something.

Aaron: Cofunctor of copoints.

Someone: I think you get like a stacky quotient and this is the coarse moduli space of the stack.

Anyway, so there's this idea in algebraic topology that étale topology should correspond to classical topology. Like for example the fundemental group-covering spaces connection works in the étale topology.

I don't want to assume you know what Grothendieck topologies are, so I'll just tell you extremely quickly. You usually think about topologies as collections of open sets, but instead you could think about collections of more general covers, not just open sets... So, like, if you have a sheaf in the étale topology, it will give you a sheaf in the classical topology by just restricting to open sets, and conversely. The sheaves are equivalent in these two categories.

So that's maybe the first idea.

The second idea is nonabelian derived functors. This should be familiar to algebraic topologists if you know about André-Quillen cohomology or the contangent complex.

We're trying to derive something like Ω^1, the collections of 1-forms on a space. The issue is that this functor, which you think of as a functor from spaces to abelian groups, is not an abelian functor, which is not a surprise because spaces is not an abelian category, so you're like, I used to know how to derive functors, but what do I do now?

The basic idea here, I think, is Dold-Kan, which says that *simplicial abelian groups* are equivalent to chain complexes. So, in general, you just want to use simplicial resolutions instead of whatever we did before.

Now let's try to derive the functor $\pi_0 : Top \to Set$. [draws dotted arrow to simplicial sets.] Great, how do we do this now? One way we could try to do this is that hopefully your space has a good cover (contractible cover and intersections are all contractible), which is probably true under... conditions.

John Pardon: If it's not nice enough you use a hypercover.

I was hoping not to say that word.

Aaron: You were going to do it anyway, or you were going to give us a hypercover that comes from a cover.

Yeah, so if I have a *good* cover $\mathcal{U} \to X$, I can take a simplicial object

$$\mathcal{U}_1 = \mathcal{U}, \quad \mathcal{U}_2 = \mathcal{U} \times_X \mathcal{U}, \quad \mathcal{U}_3 = \mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}$$

and I claim that the natural map

$$\pi_0 \mathcal{U}_\bullet \xrightarrow{\sim} X$$

is a weak equivalence.

Aaron: In the Quillen model structure on simplicial sets?

Yeah, and I've been informed that this is apparently due to Segal.

2.3

[This section is very disjointed because the discussion started moving very quickly.]

What about if X is a scheme? We could try to derive π_0 again.

I want to... define the étale homotopy type of X

$$Et(X) = lim_{\leftarrow} \pi_0 \mathcal{U}_\bullet,$$

in pro-simplicial sets.

I'd like to think further about deriving functors like this; I haven't seen people really do this too much. There's somehow not a unified way of talking about this.

Audience: No, there is. Papers of Schulman [?] .... You can define it as a universal Kan extension.

Great. It's cool that he's worked this out.

Cary: Is that his paper on enriched homotopy colimits? It involves like, middle derived functors?

Audience: Maybe.

Arnav: ... and this is this idea of pointless topology and locales and all that. But there should be some analogue of this correspondence between étale topology and classical topology.

Audience: The story should be like higher sheaf theory, and some suitable version of locally constant higher sheafs is.... I don't know this story, but someone does.

[I stopped trying to write down things that people were saying.]

Anyway, I'll just cut to the chase, since I'm running low on time:

Theorem 7. *If X is normal and proper over k, then*

$$Sym^n(X_{et}) \xrightarrow{\sim} (Sym^n X)_{et}$$

is a weak equivalence.

This is the theorem now that ties together the two parts of what I've been talking about.

This proves various things, like, there's an étale Dold-Thom theorem now. And we get all these stabilization results. I can now just say...

Theorem 8. *(Etale Dold-Thom)*

$$\pi_*^{et} Sym^\infty X \cong \tilde{H}_*^{et}(X; \mathbb{Z})$$

and now there's a good question for a roomful of topologists: why might we care?

I think this gave me a better understanding of these nonabelian derived functors, for example. And like the Weil conjectures may be viewed as point counts of these ...

[A rapid stream of speech which I couldn't keep up with.]