

PICARD GROUPS OF MODULI PROBLEMS

DANIEL LITT

[This is an exposition of Mumford’s famous paper, *Picard Groups of Moduli Problems*. Mumford’s paper was written before the notion of a stack was formalized—I will follow him by not giving a precise definition of a stack, and instead leaving some things to the reader’s imagination (or some googling). But I hope that the influence of the intervening decades will clarify some aspects of the presentation here.]

1. MODULI PROBLEMS

The following situation is ubiquitous in algebraic geometry (and indeed, throughout mathematics): one wishes to parametrize all data of a certain type, in a geometrically meaningful way. The typical way of stating this problem is as follows: one is given a functor $F : \text{Schemes} \rightarrow \text{Sets}$, and one wants to find a representing object—ideally, a scheme. Here $F(X)$ will be the collection of all data of some type on X .

Example 1. Let F be the functor sending a space X to the set of line bundles on X globally generated by $n + 1$ (chosen) global sections, that is,

$$F(X) = \{\mathcal{O}_X^{n+1} \rightarrow \mathcal{L} \rightarrow 0 \mid \mathcal{L} \text{ a line bundle}\} / \sim .$$

where $(\mathcal{O}_X^{n+1} \rightarrow \mathcal{L} \rightarrow 0) \sim (\mathcal{O}_X^{n+1} \rightarrow \mathcal{L}' \rightarrow 0)$ if there is a diagram

$$\begin{array}{ccccc} \mathcal{O}_X^{n+1} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\ \parallel & & \downarrow \cong & & \\ \mathcal{O}_X^{n+1} & \longrightarrow & \mathcal{L}' & \longrightarrow & 0 \end{array}$$

This is a functor via pullback.

Then F is represented by $\mathbb{P}^{n+1}$.

Being representable imposes strong restrictions on such an F . The most obvious is that it is a sheaf in the Zariski topology. Namely, if $\{U_i\}$ is an open cover of X , the sequence of sets

$$F(X) \rightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

is exact. But actually, representable functors are sheaves in a much stronger sense:

Theorem 1 (Faitfully Flat Descent of Morphisms). *Let $U \rightarrow X$ be a faithfully flat, quasi-compact (fpqc) morphism. Suppose F is a representable functor. Then the sequence*

$$F(X) \rightarrow F(U) \rightrightarrows F(U \times_X U)$$

is exact. [Explain what this means geometrically.]

Because of set-theoretic difficulties, we will usually work with morphisms which are faithfully flat and of finite presentation (fppf) (or even simply étale morphisms) rather than in the full generality of fpqc morphisms. Indeed, for almost all of our examples, working in the étale topology will be enough. We will thus say that a functor F is a *fpqc sheaf* (or fppf sheaf, etc.) if for any fpqc (or fppf or étale, etc.) surjective morphism $U \rightarrow X$, the sequence

$$F(X) \rightarrow F(U) \rightrightarrows F(U \times_X U)$$

is exact.

Unfortunately, many natural moduli problems are not fppf sheaves, and thus are not representable. Let us consider two (related) examples.

Example 2. Let $\pi_0(B\mathbb{G}_m)$ (I use this notation for reasons soon to be revealed) be the functor that sends a scheme X to the set of isomorphism classes of line bundles over X (note that by Hilbert’s theorem 90, I need not specify in which topology these are line bundles. This is an extremely reasonable (and important!) moduli problem. But $\pi_0(B\mathbb{G}_m)$ is not a sheaf, even in the Zariski topology—all line bundles are locally trivial in the Zariski topology, by definition, so the sheafification of $\pi_0(B\mathbb{G}_m)$ is represented by the terminal object!

Example 3. Suppose we wish to construct a moduli space $\mathcal{M}$ of elliptic curves (I’ll define this carefully later). There are some elliptic curves with automorphisms, e.g.

$$y^2 = x^3 - 1.$$

Call this curve E , and denote the point in $\mathcal{M}$ corresponding to E by e . Then by functoriality, any family of elliptic curves which is, say, étale-locally trivial and all of whose fibers are isomorphic to E must map to the point $e \in \mathcal{M}$, and thus must be isomorphic. But there are non-isomorphic families satisfying these properties over e.g. $\mathbb{G}_m$, which is a contradiction.

So, we observe that if we wish to parametrize objects with isomorphisms, we run into two (intimately related) issues:

- (1) If any of the objects X we are parametrizing admits an automorphism of finite order, there exist non-isomorphic étale-locally trivial families, all of whose fibers are isomorphic to X , over, say, $\mathbb{G}_m$. But these families would correspond to the same (constant) map to the moduli space.
- (2) There is some ambiguity in the sheaf condition—suppose we have some data X_i on U_i , $i \in \{1, 2\}$. Then if our data has automorphisms, to glue on $U_1 \cap U_2$ we must specify an isomorphism between $X_1|_{U_1 \cap U_2}$ and $X_2|_{U_1 \cap U_2}$, rather than just checking that these two restrictions are isomorphic. In the case the data has no automorphisms, this is not an issue—there is at most one choice of isomorphism.

Let us re-examine Example 2 above. I claim that if, instead of trying to classify isomorphism classes of line bundles, we try to classify all line bundles, things will work much better. Let us see in what sense line bundles can be glued together.

- (1) Descent for isomorphisms: Suppose U_i is a cover of X . If $\mathcal{L}, \mathcal{L}'$ are line bundles on X , and $f_i : \mathcal{L}|_{U_i} \rightarrow \mathcal{L}'|_{U_i}$ are isomorphisms so that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $\{f_i\}$ descends to an isomorphism $f : \mathcal{L} \rightarrow \mathcal{L}'$. (Indeed, if $g : U \rightarrow X$ is a fpqc morphism, there is an exact sequence

$$\mathrm{Isom}(\mathcal{L}, \mathcal{L}')(X) \rightarrow \mathrm{Isom}(\mathcal{L}, \mathcal{L}')(U) \rightrightarrows \mathrm{Isom}(\mathcal{L}, \mathcal{L}')(U \times_X U)$$

so $\mathrm{Isom}(\mathcal{L}, \mathcal{L}')$ is an fpqc sheaf.)

- (2) Suppose $\{U_i\}$ is a cover of X and $\mathcal{L}_i$ is a line bundle on U_i for each i . Then the data of a line bundle $\mathcal{L}$ on X restricting to each U_i is the same as an isomorphism $f_{ij} : \mathcal{L}_i|_{U_i \cap U_j} \rightarrow \mathcal{L}_j|_{U_i \cap U_j}$ for each i, j , so that the f_{ij} satisfy the obvious cocycle condition. Another way of saying this is that, letting $B\mathbb{G}_m(X)$ be the *category* of line bundles on X , with isomorphisms as morphisms, for any fpqc $U \rightarrow X$, we have that the sequence

$$B\mathbb{G}_m(X) \rightarrow B\mathbb{G}_m(U) \rightrightarrows B\mathbb{G}_m(U \times_X U) \rightrightarrows B\mathbb{G}_m(U \times_X U \times_X U)$$

is exact, in the sense of *groupoids*. [Note that this is the truncation of a cosimplicial complex; explain what this means.]

So we may say that a morphism $X \rightarrow B\mathbb{G}_m$ is the same as a line bundle on X ; but such morphisms may have isomorphisms between them.

The situation I’ve described above is essentially the definition of a “category fibered in groupoids,” which I will not define (Wikipedia or FGA Explained are both good references). But $B\mathbb{G}_m$ as we’ve described it is not yet geometric—it’s something like a functor, but we’d like a representing object, so we can do geometry. Before describing the sense in which $B\mathbb{G}_m$ is geometric, we need a little interlude on representable morphisms.

2. REPRESENTABLE MORPHISMS AND OTHER FORMALITIES

Suppose $\eta : F \rightarrow G$ is a natural transformation. Then we say η is representable if for any diagram

$$\begin{array}{ccc} & G & \\ & \downarrow \eta & \\ X & \xrightarrow{f} & F \end{array}$$

with X a scheme, the pullback exists as a scheme. The morphism η may be representable even if neither F nor G are: for example, if F is not representable, we may set $G = F \times \text{Hom}(-, Y)$ and let η be the projection. Then the pullback will be $X \times Y$.

If η is representable, and P is a property of morphisms preserved by base change, we say that η has property P if for all diagrams as above, $f^*\eta$ has property P .

The main idea will be: to do geometry with an object, it suffices to have a “nice” cover by geometric objects. This may be formalized in some situations—for example, a functor F is representable if and only if it is a Zariski sheaf and admits an open cover by schemes. Here we define an open cover in the sense above—there exists some $\{U_i\}$ and a representable morphism $\eta : \sqcup U_i \rightarrow F$ so that for any diagram

$$\begin{array}{ccc} \sqcup_i (X \times_F U_i) & \longrightarrow & \{U_i\} \\ \downarrow f^*\eta & & \downarrow \eta \\ X & \xrightarrow{f} & F \end{array}$$

we have

- (1) $\sqcup_i (X \times_F U_i)$ is a scheme, and
- (2) $f^*\eta$ is an open cover.

The notion of an algebraic space weakens this to do geometry with étale sheaves which admit an étale cover.

Remark 2. The fact that (1) holds above is a strong condition—it is useful to have a criterion for when a diagram

$$\begin{array}{ccc} & T' & \\ & \downarrow & \\ T & \longrightarrow & F \end{array}$$

admits a pullback which is a scheme, where T, T' are schemes and F is a functor. A necessary and sufficient condition is that the diagonal morphism

$$F \rightarrow F \times F$$

is representable. This is sufficient because the diagram

$$\begin{array}{ccc} T \times_F T' & \longrightarrow & T \times T' \\ \downarrow & & \downarrow \\ F & \xrightarrow{\Delta} & F \times F \end{array}$$

is a pullback diagram [\[draw picture!\]](#), and necessary because of the diagram

$$\begin{array}{ccc} T \times_{F \times F} F & \longrightarrow & T \\ \downarrow & & \downarrow \Delta \\ T \times_F T & \longrightarrow & T \times T \\ \downarrow & & \downarrow \\ F & \xrightarrow{\Delta} & F \times F \end{array}$$

We will see this criterion later.

Let us now check that $B\mathbb{G}_m$ admits an fppf (or even smooth and surjective!) cover. Namely, let pt be the terminal object, and consider the map $\text{pt} \rightarrow B\mathbb{G}_m$ defined by the trivial line bundle on pt . I claim that this is a smooth cover. To say what this means, I need to *define* the pullback

$$\begin{array}{ccc} T \times_{B\mathbb{G}_m} T' & \longrightarrow & T' \\ \downarrow & & \downarrow f' \\ T & \xrightarrow{f} & B\mathbb{G}_m. \end{array}$$

A map $X \rightarrow T \times_{B\mathbb{G}_m} T'$ should be a map into T and a map into T' , so that the compositions with f, f' agree. But $B\mathbb{G}_m(X)$ is not a set, but a category—so we say that a map into $T \times_{B\mathbb{G}_m} T'$ is a map into T, T' , and a *choice* of isomorphism of their images.

Let's work out this means. Suppose f is defined by a line bundle $\mathcal{L}$ on T and f' is defined by a line bundle $\mathcal{L}'$ on T' . Then a map $X \rightarrow T \times_{B\mathbb{G}_m} T'$ is the same as maps $g : X \rightarrow T, g' : X \rightarrow T'$, and a choice of isomorphism

$$\iota : g^* \mathcal{L} \xrightarrow{\sim} g'^* \mathcal{L}'.$$

Note that this data *does not admit any automorphisms*, so $T \times_{B\mathbb{G}_m} T'$ is an honest-to-god functor, rather than a category fibered in groupoids.

Now, let's compute the pullback

$$\begin{array}{ccc} T \times_{B\mathbb{G}_m} \text{pt} & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ T & \longrightarrow & B\mathbb{G}_m. \end{array}$$

where the bottom arrow is defined by a line bundle $\mathcal{L}$ on T . By the discussion above, a map $X \rightarrow T \times_{B\mathbb{G}_m} \text{pt}$ is the same as a map $f : X \rightarrow T$ and a map $p : X \rightarrow \text{pt}$, as well as an isomorphism $\iota : f^* \mathcal{L} \xrightarrow{\sim} p^* \mathcal{O}_{\text{pt}} = \mathcal{O}_X$. If $f^* \mathcal{L}$ is trivial, $(T \times_{B\mathbb{G}_m} \text{pt})(X) = \mathbb{G}_m(X)$; otherwise, it is empty. Furthermore, if $f_i : U_i \rightarrow T$ are maps so that $\mathcal{L}_i := f_i^* \mathcal{L}$ is trivial for all i , the automorphism of $\mathbb{G}_m(U_i \cap U_j)$ given by this identification is given by the isomorphism $\mathcal{L}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{L}_j|_{U_i \cap U_j}!$. So $(T \times_{B\mathbb{G}_m} \text{pt})$ is precisely the total space of the $\mathbb{G}_m$ -bundle associated to $\mathcal{L}$, i.e. $\text{Tot}(\mathcal{L}) \setminus \{0\}$. This is certainly faithfully flat over T .

Remark 3. It may be a surprise that the pullback here exists as a scheme, and one may ask whether this is true in general. The answer is yes, by an analogue of the criterion of Remark 2. We ask if the diagonal morphism

$$\Delta : B\mathbb{G}_m \rightarrow B\mathbb{G}_m \times B\mathbb{G}_m$$

is representable. But indeed, by our definition of pullback above, the pullback

$$\begin{array}{ccc} T \times_{B\mathbb{G}_m \times B\mathbb{G}_m} B\mathbb{G}_m & \longrightarrow & T \\ \downarrow & & \downarrow (\mathcal{L}, \mathcal{L}') \\ B\mathbb{G}_m & \xrightarrow{\Delta} & B\mathbb{G}_m \times B\mathbb{G}_m \end{array}$$

is represented by $\text{Isom}(\mathcal{L}, \mathcal{L}') = \text{Tot}(\mathcal{L}^\vee \otimes \mathcal{L}') \setminus \{0\}$.

3. QUOTIENT STACKS

One of the useful benefits of this formalism is that it allows us to define a good notion of quotient, which is in general quite subtle in algebraic geometry. A typical situation is as follows: you wish to parametrize some type of object with automorphisms. So you instead parametrize these objects along with some rigidifying data, and then “forget” the rigidifying data by quotienting out by automorphisms.

To motivate the definition that will come, consider the following situation. Suppose X is a topological space, and G is a finite group acting freely discontinuously on X , so $X \rightarrow X/G$ is a covering space. While in general we view X/G as a colimit, so it has a good “mapping out property” but no good “mapping in property,” in this case there is a good “mapping in property” which we will emulate in our discussion of quotient stacks.

Namely, (because the map $X \rightarrow X/G$ admits local sections) a map $Y \rightarrow X/G$ may be specified as follows: choose a cover U_i of Y , and maps $f_i : U_i \rightarrow X$. Furthermore, choose $g_{ij} \in G$ so that $g_{ij}f_i = f_j$, and so that the g_{ij} satisfy the cocycle condition on triple intersections.

Now suppose X is a scheme, and G is a finite group acting on X (with no condition of the action being free, or anything). In the case the action is free, and the categorical quotient $X \rightarrow X/G$ is étale, we wish to recover X/G . So imitating the above definition in the Zariski topology is not good enough, since $X \rightarrow X/G$ does not in general admit Zariski-local sections (think of e.g. finite Galois covers of a curve of genus > 0).

So we instead make the following definition: a map $Y \rightarrow X/G$ is given by an étale surjection $U \rightarrow Y$, and a map $f : U \rightarrow X$, as well as a section $g \in \Gamma(U \times_X U, G)$ so that $g\pi_1^*f = \pi_2^*f$, so that g satisfies the cocycle condition (i.e. if $\pi_{ij} : U \times_X U \times_X U \rightarrow U \times_X U$ are the projections, $\pi_{12}^*g \cdot \pi_{23}^*g = \pi_{13}^*g$). [You should convince yourself that this is analogous to the situation above.] It is not hard to check that if the categorical quotient $X \rightarrow X/G$ exists and is étale, this construction agrees with X/G . There is a natural identification of morphisms constructed by refining the étale cover.

Example 4. Suppose we are working over an algebraically closed field k , with $\text{pt} = \text{Spec}(k)$, and let G be a finite group. Let us compute the “functor of points” of $BG := \text{pt}/G$, where G acts trivially. A map $X \rightarrow BG$ is given by an étale surjection $U \rightarrow X$, a map $U \rightarrow \text{pt}$ (which is no data at all), and a G -cocycle on U —that is, precisely the data of a étale G -torsor on X ! That is, the set of isomorphism classes of morphisms $X \rightarrow BG$ is precisely the set of isomorphism classes of G -torsors, or $H^1(X_{\text{ét}}, G)$. This justifies the name BG . While $\mathbb{G}_m$ is not finite, emulating these definitions in a finer topology than the étale topology (e.g. the smooth or fppf topologies) recover $B\mathbb{G}_m$ from before as the quotient $\text{pt}/\mathbb{G}_m$.

Note that as before, $\text{pt} \rightarrow BG$ is a cover—in this case, it is an étale cover. If $T \rightarrow BG$ is a map defined by an étale G -torsor $\mathcal{G}$, the diagram

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ T & \longrightarrow & BG \end{array}$$

is a pullback square.

4. THE PICARD GROUP OF BG

The ultimate goal of this note is to compute $\text{Pic}(\mathcal{M}_{1,1})$, following Mumford. We will now do a warm-up, by defining and computing $\text{Pic}(BG)$, for a finite group G . [Can you guess what the answer is?] We work over an algebraically closed field k .

Suppose G is a finite group. Let us discuss what it means to give a line bundle on BG . If X is a scheme, and $\mathcal{L}$ is a line bundle on X , a line bundle on X is specified by giving a line bundle $\mathcal{L}$ on any fpqc (or fppf, or étale) cover $U \rightarrow X$, as well as descent data—namely, if $\pi_i : U \times_X U \rightarrow U$ are the projections, an isomorphism $f : \pi_1^*\mathcal{L} \xrightarrow{\sim} \pi_2^*\mathcal{L}$, satisfying the cocycle condition.

Remark 4. Note that we may view a line bundle as giving a lot more data than the above—namely if $\mathcal{L}$ is a line bundle on X , and $f : T \rightarrow X$ is any morphism, we get a line bundle $f^*\mathcal{L}$ on T . Furthermore, if we have a commutative triangle

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ & \searrow f' & \swarrow f \\ & X & \end{array}$$

there is a canonical isomorphism $f'^*\mathcal{L} \xrightarrow{\sim} g^*f^*\mathcal{L}$; these isomorphisms are compatible with compositions in the obvious sense.

Now, here are two (equivalent) ways of thinking of a line bundle on BG . As above, we may view such a line bundle as associating to each map $T \rightarrow BG$ (e.g. a G -torsor $\mathcal{G}$ on T) a line bundle $\mathcal{L}_{\mathcal{G}}$ —furthermore, for each $h : T' \rightarrow T$ over BG , we need to specify an isomorphism $h^*\mathcal{L}_{\mathcal{G}} \xrightarrow{\sim} \mathcal{L}_{\mathcal{G}'}$, (where $T' \rightarrow BG$ is specified by a G -torsor $\mathcal{G}'$), compatible with compositions. Note that there are automorphisms $T \rightarrow T$ over BG ,

which are the identity on T —namely, automorphisms of $\mathcal{G}$, aka G itself! So in particular, we have a map $\chi_{\mathcal{G}} : G \rightarrow \text{Aut}(\mathcal{L}_{\mathcal{G}}) = \mathbb{G}_m$ for each $\mathcal{G}$.

On the other hand, we may specify a line bundle $\mathcal{L}$ on an étale cover; note that as $k = \bar{k}$, pt has no connected étale covers, so *every* line bundle on BG comes from a line bundle and descent data on pt . Let us think about what it means to specify a line bundle with descent data on pt , which is an étale cover of BG . Of course, there is only one choice of line bundle on pt , namely $\mathcal{O}_{\text{pt}}$. Descent data is the same as an automorphism of $\mathcal{O}_G$ (here we view G as a discrete scheme), namely, a (set-theoretic) map $G \rightarrow \mathbb{G}_m$, which satisfies the cocycle condition. An unwinding of the cocycle condition, which I will omit, shows that this map is a cocycle if and only if it is a homomorphism! In particular, a line bundle is the same as a homomorphism $G \rightarrow k^*$! That is, $\text{Pic}(BG) = \text{Hom}(G, k^*) = H^1(G, k^*)$.

Note that the homomorphism $G \rightarrow k^*$ associated to a line bundle $\mathcal{L}$ is precisely the same as $\chi_{\mathcal{G}}$, where $\mathcal{G}$ is the trivial $\mathcal{G}$ -torsor over pt . We may be even more explicit: if $\chi : G \rightarrow k^*$ is a character, the total space of $\mathcal{L}_{\mathcal{G}}$ is given by

$$\text{Tot } \mathcal{L}_{\mathcal{G}} = \mathcal{G} \times_G \mathbb{A}^1$$

where G acts on $\mathbb{A}^1$ via χ .

This result should not be totally unexpected, from topology—if $k = \mathbb{C}$, note that $H^1(G, k^*) = H^2(G, \mathbb{Z})$, which precisely classifies line bundles on the (topological) space BG !

Remark 5. An identical argument shows that

$$\text{Pic}(B\mathbb{G}_m) = H^1(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z} = H^2(\mathbb{CP}^{\infty}, \mathbb{Z})(!)$$

as one might expect from the discussion above.

Remark 6. These results hold true even if k is not algebraically closed—one must apply Hilbert 90 in the argument, however.