

RATIONAL EQUIVALENCE OF 0-CYCLES, TALK

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1. INTRODUCTION

The goal of this talk is to discuss a result of Mumford from 1969, disproving a result Severi took to be essentially obvious, as late as 1934. Mumford attributes the techniques of the proof to Severi. In short, the result is this:

Let X be a smooth projective surface with $p_g > 0$. Then the group of (degree 0) Chow 0-cycles modulo rational equivalence is not “finite dimensional.” (I’ll say what this all means in a bit.)

2. RATIONAL EQUIVALENCE

Let X be a smooth, connected, n -dimensional projective scheme over a field. (Definition.) A k -cycle is a $\mathbb{Z}$ -linear combination of height $n - k$ primes of X , e.g. k -dimensional integral closed subschemes.

Of course, the group of k -cycles is huge—way too huge to be useful. To do anything with it, we need to put some equivalence relation on k -cycles; this is supposed to encode some notion of homotopy (or cobordism, depending on who you are). The relation we’ll discuss is “rational equivalence”; the (graded ring) of cycles modulo rational equivalence is called the Chow ring, denoted $CH(X)$. Here is a pithy definition:

$$CH(X) := \operatorname{coker}\left(\bigoplus_{x \in X} k(x)^* \xrightarrow{\operatorname{div}} \bigoplus_{y \in X} \mathbb{Z}\right)$$

This is a map of degree -1 , where the grading is given by the dimension of $\bar{x}$. The map div is given as follows; as a map

$$k(x)^* \rightarrow \bigoplus_{y \in X} \mathbb{Z},$$

one sends a function f to the image of its divisor on the normalization of $\bar{x}$. (Draw picture.) Let’s try to unwind this, figure out why it’s a good definition, and provide several equivalent definitions. Then we’ll specialize to 0-cycles.

An element α of $k(x)^*$ gives a flat map $\pi : \bar{x} \rightarrow \mathbb{P}^1$; the divisor corresponding to α is exactly $[\pi^{-1}(0)] - [\pi^{-1}(1)]$. Conversely, given an integral closed

$$W \subset X \times \mathbb{P}^1,$$

flat over $\mathbb{P}^1$, its image in X is an integral closed subscheme exhibiting a rational equivalence between the fiber of W over 0 and that over 1. This is the sense in which rationally equivalent k -cycles belong to a “moving family” parametrized by $\mathbb{P}^1$.

Note that there are two reasonable ways to arrive at these definitions. We require that $CH(X)$ be covariantly functorial in proper morphisms. Then rational equivalence is (1) the least relation so that any two points in $\mathbb{P}^k$ are rationally equivalent, or (2) the least relation so that any two Weil divisors associated to the same line bundle are equivalent. (Draw picture of moving families).

There is a more geometric way of viewing rational equivalence, in the case of 0-cycles—to some extent, this works more generally, but less accessibly. Namely, given a flat family $W \subset X \times \mathbb{P}^1$, whose projection to $\mathbb{P}^1$ has relative dimension 0 and whose fibers have length k , one gets a map $\mathbb{P}^1 \rightarrow \operatorname{Hilb}^k(W)$. Composing this with the Hilbert-Chow morphism $\operatorname{Hilb}^k(X) \rightarrow \operatorname{Sym}^k(X)$, where the latter parametrizes effective degree k 0-cycles on X (say what this means functorially). The relation

$$x \sim y \iff x, y \text{ are in the image of } \mathbb{P}^1$$

generates an equivalence relation on $Z_0(X)$, which coincides with rational equivalence. Explicitly, this means two 0-cycles A and B are rationally equivalent if there exists C so that $A + C, B + C$ are effective, and there

exists a map $\mathbb{P}^1 \rightarrow \text{Sym}^k(D)$ so that $A+C, B+C$ are in its image. (Draw picture of Hilbert Chow morphism etc.)

3. FINITE DIMENSIONALITY

What would it mean for $A_0(X)$ —the group of degree 0 Chow 0-cycles to be finite-dimensional? Here are four equivalent statements:

- (1) There exists some integer n so that if $\deg(A) \geq n$, A is rationally equivalent to an effective 0-cycle.
- (2) There exists n so that

$$\begin{aligned} \text{Sym}^n(X) \times \text{Sym}^n(X) &\rightarrow A_0(X) \\ (A, B) &\mapsto [A] - [B] \end{aligned}$$

is surjective.

- (3) There exists n so that for all k and $A \in \text{Sym}^k(X)$, there exists a subvariety of codimension $\leq n$ consisting of effective 0-cycles rationally equivalent to A .
- (4) (In characteristic zero) The natural map $A_0(X) \rightarrow \text{Alb}(X)(k)$ is an isomorphism.

I'll prove the first three are equivalent; the last is due to Roitman (remark on Roitman's torsion stuff).

Proof. (1) $\implies$ (2): Let n be as in (1); let $\alpha \in A_0(X)$, and let B be any effective 0-cycle of degree n . Then $\alpha + B$ is rationally equivalent to some effective 0-cycle A of degree n , so $\alpha = [A] - [B]$.

(2) $\implies$ (3): Let $x_0 \in X$. We'll soon see that

$$\{(A, B, C) \in S^n(X) \times S^n(X) \times S^m(X) \mid [A] - [B] = [C] - m[x_0]\}$$

is a countable union of closed subvarieties of the product; by (2) one of these components, say Z surjects onto $S^m(X)$; now let $W = \{C' \in S^m(X) \mid \exists (A, B), \text{ s.t. } (A, B, C), (A, B, C') \in Z\}$. (Draw picture.) Then any component of W will work.

(3) $\implies$ (1): Consider the map $S^{n-1}(X) \times X \rightarrow S^n(X)$ given by addition. This map is surjective of course—furthermore, taking n large, for any A, x let W be a co-dimension n subvariety consisting of points rationally equivalent to A . Then W has nontrivial intersection with $S^{n-1}(X) + x$, and so in particular A is rationally equivalent to $A' + x$ for some $A' \in S^{n-1}(X)$. Now if α is a zero-cycle of high degree, write it as $\alpha = [A] - [B]$ with A, B effective. Let $B = \sum [x_i]$. Then $A \sim A' + x_1$, and thus we may induct on the degree of B . $\square$

The goal of this talk is to show that none of these equivalent statements hold true of X is a smooth projective surface with $p_g > 0$, e.g. $H^0(X, \omega_X) \neq 0$.

Before doing this, let's prove the lemma I used in the implication (2) $\implies$ (3), since we'll need it later anyway.

Lemma 1. $S^n X \times S^n X$ contains a countable union of closed subvarieties Z_i so that if A and B are rationally equivalent, then (A, B) is in some Z_i . Furthermore, for each i there is a reduced scheme W_i and morphisms

$$\begin{aligned} e_i : W_i &\rightarrow Z_i \\ f_i : W_i &\rightarrow S^m X \\ g_i : W_i \times \mathbb{P}^1 &\rightarrow S^{n+m}(X) \end{aligned}$$

so that

$$\begin{aligned} g_i(w, 0) &= p_1(e_i(w)) + f_i(w) \\ g_i(w, \infty) &= p_2(e_i(w)) + f_i(w). \end{aligned}$$

with e_i surjective. (Draw picture)

Proof. Let $S_{m,k} \subset S^n(X) \times S^n(X)$ be the set of pairs (A, B) so that there exists a one-cycle $\sum n_i [E_i]$ on $S^{n+m}(X)$ with $\sum n_i \deg(E_i) = k$, $\tilde{E}_i \simeq \mathbb{P}^1$, $\cup E_i$ connected, and $A + C, B + C \in \cup E_i$.

An easy incidence-correspondence argument shows that $S_{m,k}$ is Zariski-closed. Let the Z_i be the components of the $S_{m,k}$.

Now if $(A, B) \in Z_i$, they are connected by some chain of (at most k) rational curves (draw picture). Let C_j be the intersection points of these rational curves; then letting $r : \mathbb{P}^1 \rightarrow \prod_k S^{n+m}(X)$ be the product of the embeddings of the rational curves connecting A to B , its image in $S^{k(n+m)}$ connects $A + C_1 + \dots + C_n$

to $B + C_1 + \dots + C_n$, possibly after reparametrization. Furthermore, the degree of this map onto its image is bounded in m and k .

Thus we may define

$$W_i \subset Z_i \times S^m(X) \times \text{Hom}^{\leq p}(\mathbb{P}^1, S^m(X))$$

$$W_i = \{((A, B), C, g) | g(0) = A + C, g(\infty) = B + C\}$$

where $\text{Hom}^{\leq p}(\mathbb{P}^1, S^m(X))$ denotes maps of degree at most p , and choosing p and m large, we find that W_i maps surjectively onto Z_i . Furthermore, W_i obviously admits the required maps. $\square$

4. TWO-FORMS

We will show specifically that (3) fails. The idea of the proof is to produce a non-degenerate 2-form on (most of) $\text{Sym}^n(X)$; we'll then show that any variety consisting of rationally equivalent zero-cycles must be tangent to an isotropic subspace for this form. Thus we'll bound the dimension of such varieties from above. (Remark about why this isn't surprising, and about algebraic differential equations).

First, let's produce some 2-forms. Let S be non-singular and $f : S \rightarrow S^n(X)$ a morphism, and consider the diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{f}} & X^n \\ \downarrow p & & \downarrow \pi \\ S & \xrightarrow{f} & S^n(X) \end{array}$$

where $\tilde{S} = (S \times_{S^n(X)} X^n)_{\text{red}}$. Let ω be a non-vanishing 2-form on X and $\omega^{(n)} = \sum p_i^*(\omega)$. Then $\omega^{(n)}$ is Σ_n -invariant. We now begin to work in characteristic zero.

Lemma 2. *There is a unique $\eta_f \in H^0(S, \Omega_S^2)$ so that $\tilde{f}^*(\omega^{(n)}) - p^*(\eta_f)$ is torsion.*

Proof. p is generically etale (explain the partition stratification on $S^n(X)$), so there is a large open set $S_0 \subset S$ with $p^{-1}(S_0)$ smooth. As $\tilde{f}^*(\omega^{(n)})$ is Σ_n -invariant, it descends to a unique η'_f 2-form on S_0 ; we must check that this 2-form extends (it is obviously unique). As S is smooth it suffices to check that η'_f extends to codimension 1 points of S . Let N be the normalization of $\tilde{S}$; then the pullback of η'_f to N extends to all of N , as it agrees with the pullback of $\tilde{f}^*(\omega^{(n)})$. But $\deg(p)\eta'_f$ is the trace of its pullback to N ; as we are in characteristic zero, this shows η'_f extends to η_f as desired. $\square$

Lemma 3. *η_f is functorial in f , e.g. if $f : S \rightarrow S^n(X), g : S' \rightarrow S$ $\eta_{f \circ g} = g^*(\eta_f)$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \tilde{S}' & & \xrightarrow{\tilde{f} \circ \tilde{g}} & & X^n \\ & \searrow \tilde{g} & & \nearrow \tilde{f} & \\ & & \tilde{S} & & \\ & \downarrow p_{S'} & & & \downarrow \pi \\ S' & & \xrightarrow{f \circ g} & & S^n(X) \\ & \searrow g & & \nearrow f & \\ & & S & & \end{array}$$

By uniqueness it suffices to show $p_{S'}^*(g^*(\eta_f)) - (\tilde{f} \circ \tilde{g})^*(\omega^{(n)}) = \tilde{g}^*(p_S^*(\eta_f) - \tilde{f}^*(\omega^{(n)}))$ is torsion. Thus it suffices to show that $\tilde{g}$ sends torsion differentials to torsion differentials. But indeed, any morphism of reduced f.t. schemes in characteristic zero sends torsion differentials to torsion differentials. Let $h : X \rightarrow Y$

be such a morphism, and resolve Y to create a fiber diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with Y' non-singular and $X' \rightarrow X$ dominant. Then if ω is torsion on Y , it does on Y' and thus on X' ; so $h^*(\omega)$ dies on X' . But in characteristic 0, the maps on p -forms induced by dominant morphisms are injective at height 0 points, so $h^*(\omega)$ vanishes at the generic point of each irreducible subvariety of X and is torsion. $\square$

Lemma 4. η_f is additive in f , e.g. given $f : S \rightarrow S^n(X), g : S \rightarrow S^m(X)$, denote by $f + g : S \rightarrow S^{n+m}(X)$ their sum. Then $\eta_{f+g} = \eta_f + \eta_g$.

Proof. Consider the diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\widetilde{f+g}} & X^{n+m} \\ \downarrow p & & \downarrow \\ S & \xrightarrow{f+g} & S^{n+m}(X). \end{array}$$

We wish to show that $p^*(\eta_f + \eta_g) - (\widetilde{f+g})^*(\omega^{(n+m)})$ is torsion. Note that $\omega^{(n+m)} = p_1^*(\omega^{(n)}) + p_2^*(\omega^{(m)})$ and furthermore $\widetilde{f+g} = (\tilde{f}, \tilde{g})$. Thus $(\widetilde{f+g})^*(\omega^{(n+m)}) = \tilde{f}^*(\omega^{(n)}) + \tilde{g}^*(\omega^{(m)})$, giving the claim. $\square$

5. AN ALGEBRAIC DIFFERENTIAL EQUATION

We finally come to the true content of the result.

Theorem 1 (Severi?). *Let S be non-singular and $f : S \rightarrow S^n(X)$ be a map so that all 0-cycles in the image are rationally equivalent. Then $\eta_f = 0$.*

Proof. Fix A in the image of f . By lemma 1, there exists a smooth T with a dominant map $e : T \rightarrow S$, $g : T \rightarrow S^n(X)$, $h : T \times \mathbb{P}^1 \rightarrow S^{n+m}(X)$ with

$$\begin{aligned} h(t, 0) &= f(e(t)) + g(t) \\ h(t, \infty) &= A + g(t). \end{aligned}$$

That is, T exhibits the homotopy from A to $f(e(t))$ (draw picture).

It follows that

$$\begin{aligned} \eta_h|_{T \times \{0\}} &= \eta_g + e^*(\eta_f) \\ \eta_h|_{T \times \{\infty\}} &= \eta_g. \end{aligned}$$

Now

$$\Omega_{T \times \mathbb{P}^1}^2 = p_1^* \Omega_T^2 + p_2^* \Omega_{\mathbb{P}^1}^1 \otimes p_1^* \Omega_T^1.$$

As $\Omega_{\mathbb{P}^1}^1$ has no global sections, $\eta_h = p_1^* \omega$ for some $\omega \in H^0(T, \Omega_T^2)$. In particular, η_h is constant along T and so $e^*(\eta_f) = 0$. Thus $\eta_f = 0$, as e is dominant. $\square$

We now disprove (3) from the beginning of the talk. Let $A = \sum x_i$ be a smooth point of $S^n(X)$ (e.g. all the x_i are distinct) so that ω is non-degenerate on the tangent space to each x_i . Then $\omega^{(n)}$ is non-degenerate at $(x_1, \dots, x_n)$ and $\omega^{(n)}$ is Σ_n -invariant, it descends to a non-degenerate 2-form ω' in a neighborhood of A . Let $f : S \rightarrow S^n(X)$ be a map from a smooth variety S , passing through A , and consisting only of zero-cycles rationally equivalent to A . Then $\eta_f = f^*(\omega')$ in the pre-image of a neighborhood of A ; in particular, the image of the tangent space to S at A lies in an isotropic subspace for ω' .

But ω' is non-degenerate as a symplectic form on $T_A(S^n(X))$, and thus its maximal isotropic subspaces have dimension n . So $\dim_A(\text{im}(f)) \leq n$. As the requirements on A were open conditions, we conclude that $\dim(\text{im}(f)) \leq n$. QED.