RESEARCH STATEMENT: ARITHMETIC TECHNIQUES IN ALGEBRAIC GEOMETRY

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I specialize in algebraic geometry, motivated by its rich and beautiful interactions with number theory and, in some cases, topology. My interests within algebraic and arithmetic geometry are quite broad — the unifying theme, thus far, has been the application of arithmetic techniques to more classical algebro-geometric questions. In the two years since completing my PhD, I have written about Galois actions on fundamental groups and monodromy representations, a "non-abelian" Lefschetz hyperplane theorem, the fine classification of algebraic varieties (in particular, adjunction theory), vanishing theorems via positive-characteristic techniques, algebraic dynamics, and birational geometry. The highlights of this work are marked with the symbol ($\star$) below.

My current research program is also motivated by classical questions, some of a more arithmetic bent. My projects at the moment include questions about irrationality of hypersurfaces of low degree, vanishing theorems in positive characteristic, p-adic Hodge theory, applications of fundamental group techniques to the geometric torsion conjecture, and to Iwasawa theory.

1. Overview of Past Work

1.1. **Anabelian geometry and monodromy.**

Much of my recent work has focused on Galois actions on étale fundamental groups, and on the representation theory of arithmetic fundamental groups. While I think the most exciting applications of this work are yet to come (see Section 2.1), the theory I have developed has already had significant applications to the study of monodromy representations.

The study of Galois actions on fundamental groups was initiated by Grothendieck [Gro97] and pursued by Deligne [Del89], Ihara [Iha86b, Iha86a], Anderson-Ihara [AI88, AI90], Wojtkowiak [Woj04, Woj05a, Woj05b, Woj09, Woj12], and many others. This study has largely (with the exeption of [Del89, Section 19]) been limited to an analysis of the Galois action on the pro-unipotent completion of the fundamental group, a *rational* object — motivated by the fact that this object has the structure of a mixed motive. Loosely speaking, this means that the properties of the (infinite-dimensional) Galois representation on a fundamental group are reflected in other structures on the fundamental group; for example, mixed Hodge structures (encoded in iterated integrals) or more exotic p-adic structures. This motivic philosophy of the fundamental group has been applied beautifully in arithmetic, by Minhyong Kim [Kim05], and geometry, by Hain [Hai87] and Hain-Matsumoto [HM05], for example.

In [Lit16] and [Lit17a], I began developing a theory aimed at moving beyond the pro-unipotent fundamental group, and in particular at investigating the integral structure of Galois actions on fundamental groups.

Recall that if X is a variety over a field k, with a geometric point $\bar{x}$, Grothendieck associates to $(X, \bar{x})$ a pro-finite group $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ [sga03]. If k is given an embedding into $\mathbb{C}$, this is canonically isomorphic to the pro-finite completion of the usual fundamental group of the topological space $X(\mathbb{C})^{\text{an}}$, with the analytic topology. But it additional rich structure — for example, if $\bar{x}$ comes from a k-rational point of X, $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ naturally admits an action of the absolute Galois group of k. If ℓ is a prime, we denote the pro-ℓ completion of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ by $\pi_1^{\ell}(X_{\bar{k}}, \bar{x})$.

It follows from work of Weil that Frobenius acts semisimply on $H^1(X_{\overline{\mathbb{F}_q}, \acute{e}t}, \mathbb{Q}_\ell)$ for $X/\mathbb{F}_q$ a smooth variety admitting a simple normal crossings compactification and ℓ a prime not dividing q. In [Lit16, Theorem 4.3.4], I prove a simple generalization of this result: ($\star$)

Theorem 1.1.1 (Semisimplicity of Frobenius action on π_1, [Lit16, Theorem 4.3.4])**.** *Let $X/\mathbb{F}_q$ be a smooth, geometrically connected variety admitting a simple normal crossings compactification, and ℓ a prime not dividing q. Then Frobenius acts semisimply on $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ for any rational point $x \in X(\mathbb{F}_q)$.*

Here $\mathbb{Q}_\ell[[\pi_1^{\ell,\text{ét}}(X_{\bar{k}},\bar{x})]]$ is the ring of functions on the ℓ-adic pro-unipotent completion of $\pi_1^\ell(X_{\bar{k}},\bar{x})$. This theorem is a fairly fundamental structural result (formal from the Weil conjectures in the case X is proper, but requiring geometric input in general), and seems to be new.

The action of the Galois group G_k of k on the ℓ-adic group ring $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}},\bar{x})]]$ is substantially richer, because of its integral nature; in particular, if k is finite, Frobenius does not act semi-simply, unlike the situation of Theorem 1.1.1. The rich nature of this action (analyzed in [Lit16]) places serious restrictions on the representation theory of $\pi_1^{\text{ét}}(X)$. Say a continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}) \to GL_n(\mathbb{Z}_\ell)$$

is *arithmetic* if ρ is a subquotient of a representation $\tilde{\rho} : \pi_1^{\text{ét}}(X_{\bar{k}}) \to GL_m(\mathbb{Z}_\ell)$ such that there exists a finite extension k'/k and a continuous action of $G_{k'}$ on $GL_m(\mathbb{Z}_\ell)$, compatible via $\tilde{\rho}$ with its action on $\pi_1^\ell(X_{\bar{k}},\bar{x})$. I use the analysis in [Lit16] to prove:

Theorem 1.1.2 (Theorem 1.1.2 of [Lit17a])**.** *Let X be a normal, geometrically connected variety over a finitely generated field k of characteristic zero, and let ℓ be a prime. Then there exists $N = N(X,\ell)$ such that any non-trivial, semisimple, arithmetic representation* $(\star)$

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{k}}) \to GL_n(\mathbb{Z}_\ell)$$

is non-trivial mod ℓ^N.

In many cases N is effective; for example, if $X = \mathbb{P}^1_k \setminus \{x_1, \cdots, x_m\}$, with the $x_i \in \mathbb{P}^1_k(k)$, one may show that $N(X,\ell)$ is independent of the x_i and that $N(X,\ell) = 1$ for almost all ℓ. If one assumes standard adelic open image conjectures, one has that for *any* X, $N(X,\ell) = 1$ for almost all ℓ.

Suppose X is a variety over $\mathbb{C}$. We say that a representation of the usual topological fundamental group $\rho : \pi_1^{\text{an}}(X(\mathbb{C})^{\text{an}},\bar{x}) \to GL_n(\mathbb{Z})$ *arises from geometry* if it is a torsion-free subquotient of the monodromy representation

$$\tilde{\rho} : \pi_1^{\text{an}}(X(\mathbb{C})^{\text{an}},\bar{x}) \to GL((R^i f_* \underline{\mathbb{Z}})_{\bar{x}}),$$

for some smooth proper morphism $f : Y \to X$. As a corollary of Theorem 1.1.2, one has the following fundamental consequence in classical algebraic geometry:

Corollary 1.1.3 (Immediate from Corollary 1.1.4 of [Lit17a])**.** *Let X be a normal variety over $\mathbb{C}$, and ℓ a prime. Then there exists $N = N(X,\ell)$ such that for any non-trivial representation ρ of $\pi_1^{an}(X(\mathbb{C})^{an})$ arising from geometry, one has that ρ is non-trivial mod ℓ^N.* $(\star)$

In other words, non-trivial monodromy representations on the cohomology of algebraic varieties must be non-trivial mod ℓ^N. Note that N only depends on X and ℓ, not on any auxiliary information (such as Y or the dimension of the representation in question). Theorem 1.1.2 seems to be entirely new (I am not aware of any analogous result for arithmetic representations); the independence of N from the dimension of ρ in Corollary 1.1.3 is new as well.

These results are global analogues of Grothendieck's quasi-unipotent local monodromy theorem [ST68, Appendix]. They are motivated in large part by the (geometric) torsion conjecture:

Conjecture 1.1.4 ((Geometric) torsion conjecture)**.** *Let k be a number field of degree d, (resp. the function field of a curve of a curve of gonality d), an let A/k be a (traceless) Abelian variety of dimension g. Then there exists $N = N(g,d)$ such that*

$$\#|A(k)_{tors}| < N.$$

The function field version of this conjecture has thus far been approached via complex-analytic techniques [Nad89, HT06, BT16], with only partial results in the case $g > 1$. The number field case was famously studied by Mazur [Maz77], Merel [Mer96], and many others in the case of elliptic curves ($g = 1$); essentially nothing is known for $g > 1$. The techniques used in the proof of Theorem 1.1.2 and Corollary 1.1.3 are not yet strong enough to approach the geometric torsion conjecture (see Section 2.1 for ideas in this direction), but unlike the complex-analytic techniques previously applied in pursuit of the geometric torsion conjecture, they could conceivably be applied to study the classical torsion conjecture — the case of number fields — by virtue of their anabelian nature. Moreover, the type of uniformity given by Corollary 1.1.3, while of a different nature from that of Conjecture 1.1.4, is interesting and unexpected.

1.2. **Positivity.** Much of my work in classical algebraic geometry is unified by the theme of *positivity* — in particular, given a variety X, much of my work studies the geometry of ample subvarieties of X, or ample vector bundles on X, via positive-characteristic techniques.

1.2.1. *Lefschetz Theorems.* Given a projective variety X and an ample divisor $D \subset X$, a Lefschetz hyperplane theorem indicates when the map

$$F(X) \to F(D),$$

(or $F(D) \to F(X)$, if F is covariant) is an isomorphism (or injective/surjective) where F is some contravariant functor. For example, the classical Lefschetz hyperplane theorem considers the case where $F = H^i(-, \mathbb{Z})$ or $\pi_i(-)$. In [Gro05], Grothendieck gives general techniques for answering these sorts of questions. The goal of [Lit17c], which grew out of my PhD thesis, was to extend the methods of [Gro05] to study the case where F is a *representable* functor. One application of the main result of that paper [Lit17c, Theorem 1.10] is:

Theorem 1.2.1 (Non-abelian Lefschetz hyperplane theorem [Lit17c, Theorem 1.8])**.** *Let k be a field of characteristic zero, and X a smooth projective variety over k. Let $D \subset X$ be an ample Cartier divisor, and let Y be a smooth Deligne-Mumford stack over k with Ω^1_Y nef. Then*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(D, Y) \tag{$\star$}$$

is

- *fully faithful if* $\dim(Y) = \dim(D)$*, and* $\dim(X) \geq 2$
- *an equivalence if* $\dim(Y) < \dim(D)$ *and* $\dim(X) \geq 3$.

Letting $Y = BG$, where G runs over all finite groups, one recovers the Lefschetz hyperplane theorem for the étale fundamental group; but one may also take $Y = \mathscr{M}_g$ (the moduli space of smooth genus g curves), for example, to obtain a statement about extending families of curves off of ample divisors, among many other applications. The proof of Theorem 1.2.1 uses positive-characteristic vanishing techniques; indeed, it is a special case of a more general result [Lit17c, Theorem 1.10] which works in arbitrary characteristic, and with weaker hypotheses (i.e. Ω^1_Y need not be nef). Moreover, the paper develops a toolkit aimed at proving future Lefschetz theorems (by e.g. reducing them to vanishing theorems):

The other main contribution of [Lit17c] is the development of a "Lefschetz package" — a collection of deformation-theoretic and algebraization results designed to aid in the proof of Lefschetz theorems not covered by Theorem 1.2.1 above. ($\star$)

For example, Sommese gave a conjectural classification of all smooth projective varieties X containing a $\mathbb{P}^d$-bundle as an ample divisor [BS95, Conjecture 5.5.1]. This conjecture has been the subject of much interest among those working on the fine classification of algebraic varieties; see e.g. [Som76, Fuj80, B $\breve{8}$2a, B $\breve{8}$1, B $\breve{8}$2b, BFS05, FS88, FSS87, SS86, SS90, Sat88], and see [BI09] for a survey.

In [Lit17b] I applied the "Lefschetz package" of [Lit17c] to prove many new cases of Sommese's conjecture and to reduce it in general to a conjectural characterization of projective space. This characterization was almost immediately proved by Jie Liu [Liu16], thus resolving Sommese's conjecture: ($\star$)

Theorem 1.2.2 (Theorem 1.3 of [Liu16])**.** *Let X be a smooth projective variety and $Y \subset X$ a smooth ample divisor. Suppose that $p : Y \to Z$ is a morphism exhibiting Y as a $\mathbb{P}^d$-bundle over a b-dimensional manifold Z. Then one of the following holds:*

(1) $X \simeq \mathbb{P}^3$, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *is a smooth quadric, and p is one of the projections to* $\mathbb{P}^1$.
(2) $X \simeq Q^3 \subset \mathbb{P}^4$ *is a smooth quadric threefold,* $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *is a hyperplane section, and p is a projection to one of the factors.*
(3) $Y \simeq \mathbb{P}^1 \times \mathbb{P}^b$, $Z \simeq \mathbb{P}^b$, $p : Y \to Z$ *is the projection to the second factor, and X is the projectivization of an ample vector bundle $\mathscr{E}$ on* $\mathbb{P}^1$.
(4) $X \simeq \mathbb{P}(\mathscr{E})$ *for an ample vector bundle $\mathscr{E}$ on Z, and* $\mathscr{O}_X(Y) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ *(i.e. Y is a fiberwise hyperplane).*

1.2.2. *Vanishing theorems.* One of the key tools in the application of positive-characteristic techniques to classical algebraic geometry over $\mathbb{C}$ is the use of the Frobenius morphism to increase the "positivity" of a sheaf. For example, if $\mathscr{L}$ is an ample line bundle on a projective variety X over a field k of characteristic $p > 0$, one has that for any coherent sheaf $\mathscr{F}$ on X and any $i > 0$,

$$H^i(X, \operatorname{Frob}^{*n} \mathscr{L} \otimes \mathscr{F}) = 0 \text{ for } n \gg 0,$$

by Serre vanishing, as $\operatorname{Frob}^* \mathscr{L} = \mathscr{L}^{\otimes p}$. Raynaud used this observation [DI87] to give a proof of the Kodaira-Akizuki-Nakano vanishing theorem, and this method was brought to its modern apotheosis by Arapura [Ara04, Ara06, Ara11]. Nonetheless, this work leaves open the following natural question:

Question 1.2.3. Let X be a variety over a perfect field k of characteristic $p > 0$, and let $\mathscr{E}$ be an ample vector bundle on X. Is it the case that for $\mathscr{F}$ a coherent sheaf on X, and $i \geq \operatorname{rk}(\mathscr{E})$, one has

$$H^i(X, \operatorname{Frob}^{*n} \mathscr{E} \otimes \mathscr{F}) = 0 \text{ for } n \gg 0?$$

The lower bound on i is necessary [Ara04, Example 5.9].

A positive answer to Question 1.2.3 would be extremely useful for proving vanishing theorems on singular schemes. Arapura proves that if R is a finite-type flat $\mathbb{Z}$-algebra, X is a projective R-scheme, and $\mathscr{E}$ is R-ample on X, the answer is "yes" for $(X_{\mathfrak{p}}, \mathscr{E}_{\mathfrak{p}})$ with $\mathfrak{p}$ lying in a non-empty open set of $\operatorname{Spec}(R)$ [Ara04, Theorem 6.1], but the general case appears to be open.

In [Lit17d, Theorem 2.2.1], I show that the answer to Question 1.2.3 is "yes" for varieties X admitting a flat lift X_2 to $W_2(k)$, as long as $\operatorname{rk}(\mathscr{E}) < \operatorname{char}(k)$ and $\operatorname{Frob}^* \mathscr{E}$ lifts to X_2, and I answer a variety of related questions. I also give a simple new proof of [Ara04, Theorem 6.1] (see [Lit17d, Theorem 3.0.1]), the characteristic-zero statement referenced above. $(\star)$

This result may seem to be of a rather special nature, but it is important for applications. I used the result above to prove a Manivel-type vanishing theorem for normal toric varieties, improving the well-known Bott-Danilov-Steenbrink vanishing theorem, and generalizing the main results of [BC94, Man96, BTLM97]:

Theorem 1.2.4 (Theorem 4.0.2 of [Lit17d])**.** *Let X be a normal projective toric variety over a perfect field k, and $\mathscr{E}_1, \cdots, \mathscr{E}_m$ ample vector bundles on X. Let $j : U \hookrightarrow X$ be the inclusion of the smooth locus. Then if*

(1) $\operatorname{char}(k) = 0$, or

(2) $\operatorname{char}(k) = p > \sum_i \operatorname{rk}(\mathscr{E}_i)$, and each $\mathscr{E}_i$ lifts to the canonical (toric) lift of X to $W_2(k)$, $(\star)$

then

$$H^s(X, j_* \Omega^q_U \otimes \operatorname{Sym}^{a_1} \mathscr{E}_1 \otimes \cdots \otimes \operatorname{Sym}^{a_m}(\mathscr{E}_m) \otimes \bigwedge^{b_1} \mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m} \mathscr{E}_m) = 0$$

for $s > \sum_{i=1}^m (\operatorname{rk}(\mathscr{E}_i) - b_i)$.

The bound on s is sharp, and the result should be useful for studying the cohomology of e.g. degeneration loci.

1.3. **Dynamics.** In [LL17], my collaborator John Lesieutre and I used p-adic methods to analyze automorphism groups of algebraic varieties. The main goal of this paper was to understand how the automorphism group of a variety X changes under birational modifications of X. Our main result was:

Theorem 1.3.1 (Theorem 1.5 of [LL17])**.** *Let X be a smooth projective variety over $\mathbb{C}$ of dimension n, and let $Y = \operatorname{Bl}_Z(X)$, with exceptional divisor E, where Z is a smooth subvariety of X of dimension r. If either (1) $2r + 3 \leq n$, or (2) $r + 3 \leq n$ and $\operatorname{Nef}(E)$ is a polyhedral cone, then there exists an integer $N \geq 1$ such that for any automorphism ϕ of Y, one has that ϕ^N descends to X.* $(\star)$

Case (1) of this theorem was proven in [BC13] in the case that the Picard rank of X equals 1. This result is an extension of some of the results of [Les15] to varieties of arbitrary dimension.

The paper also proves several auxiliary results of independent interest — for example, we prove a version of the dynamical Mordell-Lang conjecture for non-reduced subschemes of a given scheme [LL17, Theorem 1.1] and bound the intersection multiplicity of a divisor with its iterates under an automorphism [LL17, Theorem 7.2]; this is a generalization of a result of Arnol'd [Arn93].

1.4. **Grothendieck Ring of Varieties.** My earliest work [Lit14, Lit15] focuses on the Grothendieck ring of varieties. Let k be a field; then as an Abelian group, the Grothendieck ring of varieties over k, denoted $K_0(\mathrm{Var}_k)$, is spanned by isomorphism classes of k-varieties, subject to the relation that

$$[X] = [X \setminus Z] + [Z] \text{ if } Z \text{ is a closed subvariety of } X;$$

the multiplicative structure comes from the Cartesian product.

Motivated by the Weil conjectures, Kapranov defines the generating series

$$Z_X(t) \in K_0(\mathrm{Var}_k)[[t]], Z_X(t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)]t^n,$$

and proves that it is a rational function if X is an irreducible curve with $X(k)$ non-empty. In [Lit15], I prove that $Z_X(t)$ is rational even if $X(k)$ is empty. In [Lit14] I show that it satisfies a motivic version of the "Newton above Hodge" theorem, and analyze some consequences of this result in relation to questions of Ravi Vakil and Melanie Wood [VW15, Question 1.25].

2. Current and Future Work

2.1. **Anabelian geometry and applications.** My research program at the moment has grown out of the work described in Section 1.1, and I envision applications far beyond those described there. The program has several components, each with different possible applications — the main idea is to use recent developments in integral p-adic Hodge theory to compute and exploit Galois actions on fundamental groups.

2.1.1. *p-adic methods: around the geometric torsion conjecture.* Theorem 1.1.2 is stated over finitely-generated fields of characteristic zero; however, I have shown that in some cases it holds over other fields. For example, I have, in unpublished work, proven:

Theorem 2.1.1 (Litt, unpublished). *Let K be a p-adic field and $X = \mathbb{P}^1_K \setminus \{x_1, \cdots, x_m\}$, for $x_1, \cdots, x_m \in \mathbb{P}^1_K(K)$. Then there exists $N = N(X)$ such that any non-trivial semisimple arithmetic representation*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{K}}) \to GL_n(\mathbb{Z}_p)$$

is non-trivial mod p^N.

The proof uses some mild integral p-adic Hodge theory. The main benefit of the proof is the following observation: if X has good reduction, the constant N in Theorem 2.1.1 may be bounded from above in terms of certain invariants of X — so-called p-adic iterated integrals, originally constructed by Coleman [Col82]. These are the p-adic analogue of complex (iterated) line integrals [Che71]. The key step in proving such a bound (which is not required for the proof of Theorem 2.1.1 itself) is the following comparison result (which I call a conjecture, as not all the details have been written down yet — I expect I will have a complete proof soon):

Conjecture 2.1.2. *Let K be a p-adic field and X/K a smooth K-variety. Suppose there exists $\overline{X}/\mathscr{O}_K$ proper, flat, reduced and containing X, such that*

(1) $\overline{X}$ is regular with reduced special fiber, and
(2) $\overline{X} \setminus X$ has simple normal crossings.

Then:

(1) For any rational basepoint of X, there is a natural Galois-equivariant and Frobenius-equivariant isomorphism of pro-unipotent B_{st}-group schemes $\pi_1^{dR,unip}(X)\widehat{\otimes}B_{st} \xrightarrow{\sim} \pi_1^{\acute{e}t,unip}(X)\widehat{\otimes}B_{st}$ preserving filtrations, G_K-actions, Frobenius actions, and monodromy; here $\pi_1^{dR,unip}$ is the unipotent de Rham fundamental group.
(2) There is a natural cocommutative $\mathscr{O}_K$-Hopf algebra $\mathscr{O}_{\pi_1^{dR}}$, complete with respect to its augmentation ideal, such that $K\widehat{\otimes}\mathscr{O}_{\pi_1^{dR}}$ is the Hopf algebra underlying $\pi_1^{dR,unip}$ (i.e. $\pi_1^{dR,unip}(X)$ has a natural integral structure, depending on $\overline{X}$). There is a natural comparison map

$$\mathscr{O}_{\pi_1^{dR}}/\mathscr{I}^n \otimes_{R_{\mathscr{O}_K}} \widehat{A_{st}} \to \mathbb{Z}_p[[\pi_1^p(X_{\bar{K}})]]/\mathscr{I}^n \otimes_{\mathbb{Z}_p} \widehat{A_{st}},$$

where $\mathscr{I}$ denotes the augmentation ideal on both sides, whose kernel and cokernel are annihilated by an explicit (in terms of n and $\dim(X)$) element of $\widehat{A}_{st}$.

(3) If $(X, \overline{X})$ has good reduction, one may replace $B_{st}, \widehat{A_{st}}$ with B_{crys}, A_{crys} in the statements above.

Loosely speaking, this conjecture gives an *integral* comparison theorem between the de Rham fundamental group of X and its pro-p étale fundamental group. The period rings in the statement above are defined as in [Bha12]; their precise definitions are not important for the purposes of this research statement. This conjecture generalizes the comparison result [AIK15, Theorem 1.4].

Question 2.1.3. In [BMS15], the authors define an integral p-adic cohomology theory specializing to both étale and de Rham cohomology (among others), called A_{inf}-cohomology (for smooth proper varieties of good reduction). Is there a natural A_{inf}-fundamental group?

I expect that the methods of the proof of Pre-Theorem 2.1.2 will give a positive answer to this question for smooth proper varieties of good reduction; for the best possible applications, however, I require the statement for arbitrary smooth varieties.

The key step in applying Conjecture 2.1.2 to bound the constant N of Theorem 2.1.1, and to generalize that theorem to curves of genus greater than 0, is an explicit answer to the following question:

Question 2.1.4. Can one bound the p-adic valuation of p-adic iterated integrals on $X \setminus D$, where X is a smooth proper curve of good reduction, and $D \subset X$ is a divisor of good reduction?

I expect that this is doable, at least in the case X has genus zero. More ambitious is the following question:

Question 2.1.5. Can one use Conjecture 2.1.2 to generalize the theory of p-adic iterated integration to semi-stable curves? If so, can one bound their p-adic valuations?

I am optimistic regarding a positive answer to this question, though it is very much work in progress. A strong enough answer to Question 2.1.5 would imply, by the methods of the proof of Theorem 2.1.1, an answer to the geometric torsion conjecture (Conjecture 1.1.4 in the function field case), which is the ultimate goal of this project.

2.1.2. *p-adic methods: Iwasawa theory.* Another expected application of Conjecture 2.1.2 is in Iwasawa theory. First, I hope to recast Rubin's proof of the Iwaswa main conjecture for cyclotomic fields (appearing in [Lan90, Appendix], and using cyclotomic units) in terms of the Galois action on the fundamental group(oid) of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$; this project is nearly complete. I expect that the existence of such a proof is interesting but not completely unexpected to experts (see e.g. [NSW17] for a proof of certain explicit reciprocity laws via fundamental group techniques).

A more ambitious project would be to recast Rubin's proof of the main conjecture for imaginary quadratic fields [Rub91] in terms of the Galois action on the fundamental group of $E \setminus \{0\}$, where E is a well-chosen CM elliptic curve.

More generally, Conjecture 2.1.2 seems to yield explicit reciprocity laws not yet in the literature, via judicious choices of X. Already taking $X = \mathbb{P}^1_{\mathbb{Q}_p} \setminus \{0, 1, \infty\}$ gives an (infinite-dimensional) representation of $G_{\mathbb{Q}_p}$ which may be described fairly explicitly in terms of so-called p-adic multiple L-values (see [Fur04, Fur07] for the definition); this explicit description is an explicit reciprocity law in that it specializes to e.g. the explicit recprocity law of [NSW17]. The p-adic multiple zeta values arising in this description may be interpolated [FKMT17], giving multiple p-adic L-functions.

Question 2.1.6. Is there a "Iwasawa main conjecture" for these multiple p-adic L-functions?

I believe the answer to be "yes" — that is, if $\Lambda = \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}(\mu_p^\infty)/\mathbb{Q})]] = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, I can construct certain natural finite-dimensional modules over $\Lambda^{\otimes m}$, defined purely Galois-theoretically, whose support appears to be controlled by multiple p-adic L-functions — the case $m = 1$ is the usual Iwasawa main conjecture for $\mathbb{Q}$. At present this is very much at the level of a conjecture, but it certainly seems a promising avenue of inquiry.

2.1.3. *ℓ-adic methods.* The p-adic methods described in the previous two sections promise to elucidate the actions of Galois groups of p-adic fields on pro-p fundamental groups (in terms of p-adic Hodge-theoretic data, i.e. p-adic iterated integrals). This local data is not the complete picture, of course.

Question 2.1.7. Let $\mathbb{F}_q$ be a finite field and ℓ a prime not dividing q. Let $X = \mathbb{P}^1_{\mathbb{F}_q} \setminus \{x_1, \cdots, x_m\}$ for $x_i \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_q)$. How does Frobenius act on $\pi_1^\ell(X_{\overline{\mathbb{F}_q}}, \bar{x})$, where $\bar{x}$ is the geometric point associated to a rational point of X?

Of course this question is interesting for any X, though it seems most approachable in the case of genus zero curves, via Anderson-Ihara theory [AI88, AI90]. More precisely, the following question seems to be accessible:

Question 2.1.8. Let X be as in Question 2.1.7. Let

$$\epsilon : \mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{\mathbb{F}_q}}, \bar{x})]] \to \mathbb{Z}_\ell$$

be the augmentation map, and $\mathscr{I} = \ker(\epsilon)$ the augmentation ideal. What is the image of

$$(\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{\mathbb{F}_q}}, \bar{x})]]/\mathscr{I}^n)^{\mathrm{Frob}=q}$$

under the augmentation map?

This is a precise ℓ-adic analogue of the p-adic Question 2.1.4, and has analogous applications to the geometric torsion conjecture. I have already studied this question in [Lit16] and [Lit17a]; related questions have been studied by Deligne [Del89, Section 19] and Wojtkowiak [Woj04, Woj05a, Woj05b, Woj09, Woj12]. I hope to give a precise answer in joint work with Jeremy Booher and Brian Lawrence.

2.2. **Birational geometry.** In joint work with Alex Perry, I am studying several questions relating specialization of rationality, and to the rationality of hypersurfaces and, more generally, complete intersections, motivated by the recent work of Nicaise-Shinder [NS17] and Kontsevich-Tschinkel [KT17]. The main result of Kontsevich-Tschinkel is that rationality specializes in smooth families, in equicharacteristic zero.

Question 2.2.1. Does rationality specialize in mixed or positive characteristic?

The techniques of [NS17, KT17] rely on the weak factorization theorem, which is not known to hold in positive or mixed characteristic; nonetheless, we have some ideas of how to approach this question.

More concretely, [KT17] provides us with a birational invariant which we expect to be extremely useful in proving non-rationality results. For example, we have already proven:

Theorem 2.2.2 (Litt, Perry)**.** *A very general quartic hypersurface in* $\mathrm{Gr}(2,5)$ *is not rational.*

Our method gives us interesting implications between rationality results for hypersurfaces of different degrees and dimensions, e.g. the following surprising theorem:

Theorem 2.2.3 (Litt, Perry)**.** *Suppose that no smooth cubic threefold is stably rational. Then a very general quintic fivefold is not stably rational.*

We hope to use these techniques to study irrationality of hypersurfaces and complete intersections. For example, we hope to improve the state of knowledge on the following question:

Question 2.2.4. What is the least $a \in \mathbb{R}_{\geq 0}$ such that there exists a sequence of positive integers (d_i, n_i), with $n_i \to \infty$ and $\frac{d_i}{n_i} \to a$, such that there exists a smooth non-rational hypersurface of degree d_i in $\mathbb{P}^{n_i}$?

One expects that a very general cubic hypersurface in $\mathbb{P}^n$ is not rational as long as $n > 3$; under this expectation one may take $a = 0$ above. However, the best-known value for a is $a = \frac{2}{3}$ [Kol95, Tot16]. In principle, our method has the potential to improve this state of affairs to $a = \frac{1}{2}$, though this hope relies on several as-yet-incomplete steps. The current test case we are considering is:

Question 2.2.5. Is a very general degree 8 hypersurface in $\mathbb{P}^{12}$ stably rational?

This result seems to be accessible to our techniques, and is just outside the range of known results (we hope to be able to prove that the answer is “no”). As in [HPT16, HPT17], we attempt to reduce to the case of certain quadric bundles (though in a non-obvious way). This reduction step is complete; it remains to show the relevant quadric bundles are not stably rational, for which many tools exist (for example, decomposition of the diagonal and further degeneration to varieties with unramified cohomology, as in e.g. [Sch17]). These tools are at present more an art than a science, but our partial results are promising.

2.3. **Positivity.** My main project on positivity, building on the work described in Section 1.2, is an attempt to answer Question 1.2.3. Perhaps surprisingly, this question may be approached almost entirely via the representation theory of the algebraic group GL_n in positive characteristic — indeed, the key insights of the proof of [Ara04, Theorem 6.1] and [Lit17d, Theorem 2.2.1], which both give partial answers to Question 1.2.3, are representation-theoretic. The key question is:

Question 2.3.1. Let k be a field of characteristic $p > 0$, and let $\mathrm{Rep}_k(GL_n)$ be the category of algebraic representations of $GL_{n,k}$. Is there a representation of the Frobenius-twist functor

$$\mathrm{Frob}^{*m} : \mathrm{Rep}_k(GL_n) \to \mathrm{Rep}_k(GL_n),$$
$$V \mapsto V^{(p^m)}$$

by Schur functors, with length bounded above by n?

Arapura observes in the course of the proof of [Ara04, Theorem 6.1] (where he gives a partial answer to Question 1.2.3) that the answer is “yes” if $m = 1$; the general case appears to be unknown. In [Lit17d], I observe that much less is needed too prove Arapura’s theorem and many related results — one only requires that many of the differentials in a certain spectral sequence vanish.

This leads one to consider the structure of the derived category $D^b(\mathrm{Rep}_k(GL_n))$, which appears implicitly in [Lit17d] — showing that certain objects in $D^b(\mathrm{Rep}_k(GL_n))$ have no non-zero morphisms between them would give new cases in which Question 1.2.3 has a positive answer.

Question 2.3.2. Can one give an explicit description of $D^b(\mathrm{Rep}_k(GL_n))$?

This question is likely too ambitious, though much beautiful work has been done, in e.g. [FFSS99]. One hope for progress is to use Cartier theory for the stack BGL_n, and for the morphism $[\mathbb{A}^n/GL_n] \to BGL_n$, to study $D^b(\mathrm{Rep}_k(GL_n))$; this idea is implicit in [FFSS99], for example, and explicit in [Lit17d] and [Tot17].

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