STABLE HOMOTOPY GROUPS OF SPHERES VIA AD HOC METHODS

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1. Introduction

The goal of this talk is to compute the first couple stable homotopy groups of spheres, via strictly ad hoc methods. At some point (π_3^s) I won't be able to do this anymore, so I'll turn the talk into a dsicussion, with maybe half an idea of how to compute π_3^s.

The main tool here will be the Barratt-Priddy-Quillen Theorem, which asserts that the following four spaces are weakly equivalent:

(1) $\Omega^\infty S^\infty := \varinjlim \Omega^n S^n$, where the maps are given by the adjunciton of Σ and Ω (aka QS^0);
(2) $(\mathbb{Z} \times \Sigma_\infty)_+$, where Σ_∞ is the group of automorphisms of a countable set having finite support, and $+$ is the Quillen $+$ construction, which I'll discuss in a bit.
(3) The group completion of $B\left(\coprod_n \Sigma_n\right)$. Here $\coprod_n \Sigma_n$ is the category whose objects are the integers, with $\mathrm{Hom}(m,n) = \emptyset$ if $m \neq n$ and $\mathrm{Hom}(n,n) = \Sigma_n$. This category is a monoid under the concatenation map $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$; it is this operation with respect to which we group complete.
(4) $K(\mathrm{FinSet}) := \Omega|S^\bullet.\,\mathrm{FinSet}\,|$, where this last is the Waldhausen $S^\bullet$ construction and FinSet is the category of pointed finite sets. I won't say too much about this unless people really want me to.

I'll give a proof of this theorem from Segal's paper "Categories and Cohomology Theories." The non-technical content is in proving that the first space $\Omega^\infty S^\infty$ is weakly equivalent to any of the others—while the other equivalences are not easy (in particular, (2) $\simeq$ (3) is basically the group completion theorem) and are far from contentless, they are boring. Also, everything I'll need will be contained in the equivalence (1) $\simeq$ (2).

After computing $\pi_0^s, \pi_1^s, \pi_2^s$ using the Barratt-Priddy-Quillen theorem and some group theory, I'll discuss a relationship between homology spheres and the homotopy groups of spheres.

2. The Barratt-Priddy-Quillen Theorem

We begin with some discussion of a certain model for topological monoids, which Segal calls Γ-spaces. Here Γ is the following category:

- The objects are all finite sets.
- $\mathrm{Hom}(S,T) = \{\theta : S \to 2^T \mid \theta(i) \cap \theta(j) = \emptyset \text{ if } i \neq j\}$.
- Composition is given by $\theta \circ \psi(i) = \bigcup_{x \in \psi(i)} \theta(x)$.

A Γ-space is a contravariant functor $A : \Gamma \to \mathrm{Top}$ so that

- $A(\emptyset)$ is contractible
- The map $A([n]) \to A([1]) \times \cdots \times ([1])$ (induced by the maps $i_k : 1 \mapsto \{k\}$, $k \in [n]$) is a homotopy equivalence.

This should be thought of as a monoid as follows—the "underlying set" is $A([1])$, and $A([n])$ is identified with n-tuples of elements. Then the map $A([n]) \to A([1])$, "multiplication" is induced by the map $1 \mapsto \{1, ..., n\} \subset 2^{[n]}$. This in fact turns $A(1)$ into a particularly nice H-space.

Note that Γ spaces may be thought of as simplicial spaces, through the natural embedding $\Delta \to \Gamma$. In particular, we may take geometric realizations. Note also that "level 1" of $|A|$ is homotopy equivalent to $SA(1)$.

If A is a Γ-space, we may set $BA(S) = |T \mapsto A(S \times T)|$. I will leave checking that this is a Γ-space as a tedious exercise. Note that $BA(1) = |A|$. In particular, there is a natural map from $SA(1) \to |A| \simeq BA(1)$, so the $B^k A(1)$ form a (pre)-spectrum; call the functor associating to a Γ-space a pre-spectrum $\mathbf{B}$.

Now if $\mathcal{C}$ is a category in which sums exist (note that they exist only up to isomorphism!), we may use it to construct a Γ-space as above. Let S be a finite set, and let $\mathcal{P}(S)$ be the category of subsets of S, with maps given by inclusion. Let $\mathcal{C}(S)$ be the following category:

- The objects are functors $\mathcal{P}(S) \to \mathcal{C}$ which take disjoint unions to sums;
- The morphisms are natural *isomorphisms*.

Then $\Gamma_{\mathcal{C}} : S \mapsto |\mathcal{C}(S)|$ is a Γ-space. If $\mathcal{S}$ is the skeleton of the category of finite sets,

$$|\Gamma_{\mathcal{S}}(1)| = B\left(\coprod_n \Sigma_n\right)$$

Now, we have a way $\mathbf{B}$ of associating to each Γ-space a spectrum; let us see that we have a way of associating to each (Ω)-spectrum a Γ-space.

Now for each spectrum X, we define the Γ space $\mathbb{A}X(n) = \mathrm{Mor}(\mathbb{S}^n, X)$. $\mathbb{A}$ is right adjoint to $\mathbb{B}$ in the category Γ of Γ-spaces with level-wise acyclic Hurewicz fibrations inverted.

I'll provide one of the adjoint maps, $\mathbf{BA}(X) \to X$. For each m, n, we must give

$$(\mathbb{S}^n)^m \times \mathrm{Mor}(\mathbb{S}^m, X) \to (\omega X)_n;$$

this is given by evaluation.

Now we may prove the Barratt-Priddy-Quillen theorem, which we'll prove as the equivalent statement:

Theorem 1. $\mathbf{B}\Gamma_{\mathcal{S}} \simeq \mathbb{S}$.

Proof. By Yoneda, it is enough to give a natural isomorphism

$$\mathrm{Hom}_{Sp}(\mathbf{B}\Gamma_{\mathcal{S}}, X) \simeq \mathrm{Hom}_{Sp}(\mathbb{S}, X) \simeq \pi_0(X).$$

By adjointness, it suffices to show

$$\mathrm{Hom}_{\Gamma}(\Gamma_{\mathcal{S}}, \mathbf{A}(X)) = \pi_0(X);$$

furthermore, $\pi_0(X) = \pi_0(\mathbf{A}X(1))$ by construction.

There is a natural map, given by looking at the component containing the image of the point $B\Sigma_1 \subset \Gamma_{\mathcal{S}}(1)$ in $\mathbf{A}X(1)$. Let's construct an inverse.

To do this, we'll need an auxiliary construction. Let F be a functor from the category of finite sets with inclusions to Top, and let $\mathcal{S}_F$ be the topological category whose objects are pairs $(S, x \in F(S))$ and whose morphisms are maps $\phi : S \to T$ so that $F(\phi)(x) = y$. Then one can form $\Gamma_{S,F}$ as before; this is a Γ-space if $F(S \sqcup T) \simeq F(S) \times F(T)$.

Now choose $a \in AX(1)$, and let F_n be the homotopy fiber of the map $\mathbf{A}X(n) \to \mathbf{A}X(1) \times \cdots \times \mathbf{A}X(1)$ at the point $(a, ..., a)$; this is contractible by the definition of a Γ-space; $n \mapsto F_n$ is a functor as before. There is obviously a map $\Gamma_{F,S} \to AX$.

But by contractibility of the F_n, the (forgetful) map $\Gamma_{S,F} \to \Gamma_F$, induced by sending (S, x) to S is an isomorphism in Γ. This proves the theorem. $\square$

Now applying ω gives $\Omega BB\left(\coprod_n \Sigma_n\right) \simeq \Omega B|\Gamma_{\mathcal{S}}(1)| = \Omega^\infty S^\infty$. The term on the left is the group completion of $B\left(\coprod_n \Sigma_n\right)$, giving BPQ.

Now the group completion theorem implies that the group completion of $B\left(\coprod_n \Sigma_n\right)$ has identity component with the same homology as $B\left(\coprod_n \Sigma_n\right)$, but with abelianized fundamental group (that is $\mathbb{Z}/2\mathbb{Z}$ rather than Σ_∞). Now, for any CW complex X and a perfect normal subgroup $N \subset \pi_1(X)$, one may associate a space X^+ with a map

$$q_N : X \to X^+$$

such that

- $\ker(\pi_1(q_N)) = N$
- The homotopy fiber of q_N is acyclic (in particular, q_N is a homology equivalence).
- q_N is homotopy-universal for maps $f : X \to Y$ with $N \subset \ker(\pi_1(f))$.

I'll describe a construction of this space in a bit—but in particular, note that $q_{A^\infty} : B\left(\coprod_n \Sigma_n\right) \to \Omega BB\left(\coprod_n \Sigma_n\right)$ exhibits $\Omega BB\left(\coprod_n \Sigma_n\right)^+$ as $B\left(\coprod_n \Sigma_n\right)^+$ by Whitehead.

Now, we construct X^+ in the most naive way possible. As N is perfect, it maps to zero in H_1; choose a set of generators. Kill them by attaching 2-cells; in H_2 this introduces a $\mathbb{Z}$ for each relation between the generators—this doesn't alter H_1 by the perfection of N; however, these can be chosen so there are no syzygies, so one may fix the homology by only adding 3-cells. This can easily be seen to be universal—any map from $X \to Y$ killing N extends to the new 2-cells as the map kills N; of course it similarly kills the relations so the map extends to the 3-cells.

3. Stable Homotopy Groups

We're ready to start computing. Manifestly

$$\pi_0^s = \mathbb{Z}.$$

By the + construction description,

$$\pi_1^s = \Sigma_\infty^{ab} = \mathbb{Z}/2\mathbb{Z}.$$

Now note that $\widetilde{B\Sigma_\infty}$ is homology-equivalent to the classifying space of the alternating group, and thus $\widetilde{B\Sigma_\infty}^+ \simeq BA_\infty^+$. In particular

$$\pi_2^s \simeq H_2(A_\infty, \mathbb{Z}).$$

So let's discuss this group a little bit. Because of the short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$$

and

$$0 \to \mathbb{C}^* \to GL(n, \mathbb{C}) \to PGL(n, \mathbb{C}) \to 0$$

this group measures the failure of projective representations to lift to honest-to-god representations. In particular (by e.g. inflation-restriction), $H_2(G, \mathbb{Z})$ is the kernel of a central extension

$$0 \to H_2(G, \mathbb{Z}) \to \hat{G} \to G \to 0$$

where $\hat{G}$ is the so-called universal covering group of G—namely, a group to which projective reps of G lift uniquely. One way to construct such groups is to take a faithful projective rep of G, and pull back to GL; then for large enough representations (if G is finite) this will contain the univeral covering group.

For A_n with large n, one may take the permutation representations $S_n \to GL(n, \mathbb{R})$, and let V be the larger irreducible sub-rep. This gives a map $S_n \to O(n-1)$, which restricts to a map $A_n \to SO(n-1)$. Pulling back to Spin exhibits the requisite covering, and the kernel is $\mathbb{Z}/2\mathbb{Z}$.

Let's talk about the higher stable homotopy groups now. If M is a homology sphere of dimension n with homotopy group $\pi_1(M)$ (which is necessarily perfect, by Hurewicz), then any action of $\pi_1(M)$ on a finite set gives an element of π_n^s as follows. Namely, there is a natural map

$$M \to K(\pi_1(M), 1)$$

(which can be viewed either as classifying the universal cover of M or via building $K(\pi_1(M), 1)$ by killing higher homotopy groups of M. Now a map $\pi_1(M) \to \Sigma_N$ gives a map $K(\pi_1(M), 1) \to K(\Sigma_N, 1)$; the composition gives a map

$$S^n \to \Omega^\infty S^\infty$$

after applying the + construction. (Open to audience participation)